13 March 2008

E. Frenkel — **Langlands correspondence for loop groups**

Joint work with D. Gaitsgory (8 papers)

Reviewed in book with same title.

1) **classical setup**: Local Langlands correspondence.

F - local non-archimedean field, such as $\underbrace{\mathbb{F}_q((t))}_{\text{focus on this}}$ or a finite extension of $\mathbb{Q}_p$

Parameterize irreducible representations of $GL_n(F)$, smooth reps: $\forall v \in V, \exists N \in \mathbb{Z}_+$ s.t. $K_N \cdot v = v$

$$GL_n(F) \supset K_N = \{ g \in GL_n(\mathcal{O}) \,/\, g \equiv 1 \mod t^N \}$$

$$\begin{array}{ccc} \mathcal{O} & = & \mathbb{F}_q[[t]] \\ \cap & & \cap \\ F & = & \mathbb{F}_q((t)) \end{array}$$

$$\left\{\begin{matrix}\text{irreducible}\\ n\text{-dim reps}\\ \text{of } \cancel{\mathrm{Gal}(\bar F/F)}\ W_F\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{irred. smooth cuspidal}\\ \text{reps of } GL_n(F)\end{matrix}\right\}$$

$$\begin{array}{ccc} \mathrm{Gal}(\bar F/F) & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \\ \cup & & \parallel \\ & & \hat{\mathbb{Z}} \\ & & \cup \\ W_F & \longrightarrow & \mathbb{Z} \end{array}$$

Galois group

Weil group

$GL_n \longrightarrow G$ - reductive algebraic group, split over F

$$\{ W_F \to {}^L G \} \quad \longleftarrow\!-\!-\!\longrightarrow \quad \left\{\begin{matrix}\text{irred. smooth}\\ \text{reps of } G(F)\end{matrix}\right\}$$

${}^L G$ - Langlands dual group

② Replace $\mathbb{F}_q \longrightarrow \mathbb{C}$

$F = \mathbb{F}_q((t)) \longrightarrow \mathbb{C}((t))$

$G(F) \longrightarrow G(\mathbb{C}((t))) = G((t))$ - formal loop group

reductive complex Lie group

~~Irred smooth reps. of $G((t))$~~

<u>Lemma</u> Irred smooth reps of $G((t))$ on a $\mathbb{C}$-vector space is neccessarily trivial.

Using "matrix elements", can realize reps. of $G(F)$ in functions on $G(F)/K$, K-compact subgroup such as K_N.

$G(F)/K$ - set $\mathbb{F}_q$-points of an ind-scheme / $\mathbb{F}_q$

$K = K_0 = G(\mathcal{O})$, affine Grassmannian

Y- scheme$/\mathbb{F}_q$, $\mathcal{F}$- "ℓ-adic sheaf" on Y then get a function on $Y(\mathbb{F}_q)$ - the set of $\mathbb{F}_q$-pts of Y

$y \in Y(\mathbb{F}_q) \longmapsto Tr(Fr_y, \mathcal{F}_y)$

stalk of $\mathcal{F}$ at y.

$G(F) \circlearrowright$ <u>vector space</u> of functions on $G(F)/K$ $\xleftarrow{Tr(Fr,\,\cdot)}$ <u>category</u> ℓ-adic perverse sheafs, on an ind-scheme such as affine Grassmannian$/\mathbb{F}_q$ $\circlearrowleft G(F)$

<u>categorification</u>

$\downarrow$

$\mathbb{C}$ - perverse sheafs on these ind-schemes $/\mathbb{C}$. $\circlearrowleft G((t))$

$\xrightarrow{\text{Riemann Hilbert correspondence}}$ (twisted) $\mathcal{D}$-modules on Gr $\circlearrowleft G((t))$ $\longrightarrow$ representations of $\hat{\mathfrak{g}}$ $\circlearrowleft G((t))$

$\uparrow$ analogue of Beilinson-Bernstein equivalence.

$$\mathcal{D}\text{-mod}(G/B) \xleftrightarrow{\text{equivalence}} \mathfrak{g}\text{-mod}$$

$\uparrow$ flag variety. $\quad$ $\uparrow$ fixed central character

$$\mathcal{F} \longmapsto \Gamma(G/B, \mathcal{F})$$

<u>Upshot</u>: need to classify representations of $G((t))$ on categories (abelian) and to express "the parameter space" in terms of the Langlands dual group ${}^L G$

<u>Natural candidates</u>:

- (twisted) $\mathcal{D}$-modules on Gr and such
- representations of $\hat{\mathfrak{g}}$

3) Langlands parameters

classically, $Gal(\bar{F}/F) \longrightarrow {}^L G$

$F = k(X)$ $\qquad$ $Gal(\bar{F}/F) \approx \pi_1^{et}(X) \leftarrow$ or $\pi_1^{et}(X \setminus \{x_1, \dots, x_n\})$

X - alg. var. $/k$ $\qquad$ $k(Y)/k(X)$

$\{\pi_1(X \setminus \{x_1,\dots,x_n\}) \longrightarrow {}^L G\} \longrightarrow \{(\mathcal{F}, \nabla),$ $\mathcal{F}$ - principal ${}^L G$-bundle on X, ∇ - flat connection w/ poles $\}$

arbitrary.

In our case, $F = \mathbb{F}_q((t)) \rightsquigarrow \mathbb{C}((t))$

$X = D^\times := \operatorname{spec} \mathbb{C}((t))$

Langlands params. in our case

$= \{ (\mathcal{F}, \nabla),$ $\mathcal{F}$ - prin. ${}^L G$-bundle on $D^\times$, ∇-connection $\}$

Trivialize $\mathcal{F}$. $\qquad$ $\nabla = \partial_t + A(t)$, $A(t) \in {}^L\mathfrak{g}((t))$

t-coord. on $D^\times$

change triv. by $g(t) \in {}^LG((t))$ $\longmapsto$ $\nabla \longrightarrow \partial_t + gAg^{-1} - \cancel{\partial_t g}\,\partial_t g^{-1} g$ - gauge tr

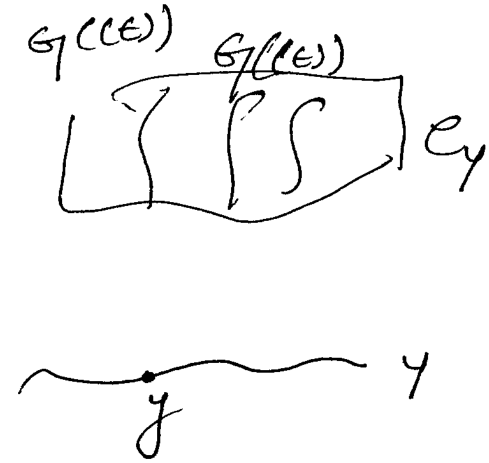
Langlands param $= \left\{ \partial_t + A(t),\ A(t) \in {}^L\mathfrak{g}((t)) \right\} \Big/ {}^LG((t))$

$\quad\Vert \quad = \mathrm{Loc}_{{}^LG}(D^*)$

$\sigma \longmapsto \mathcal{C}_\sigma$ - a category w/ an action of $G((t))$

Moreover, $\mathrm{Loc}_{{}^LG}(D^*)$ should be the universal parameter space for categorical reps. of $G((t))$.

Any category over some base Y equipped w/ a fiberwise action of $G((t))$ should neccessarily come by pull-back from a universal category under some map.



$$\begin{tikzcd}
\mathcal{C}_Y \arrow[r] \arrow[d, "G((t))\ \circlearrowright"'] \arrow[dr, phantom, "\square"] & \mathcal{C}_{univ} \arrow[d, "\circlearrowright"'] & \mathcal{C}_\sigma \arrow[l] \arrow[d] \\
Y \arrow[r] & \mathrm{Loc}_{{}^LG}(D^*) \ni \sigma & y = pt \arrow[l]
\end{tikzcd}$$

We have constructed a candidate for $\mathcal{C}_{\mathfrak{g},Y}$ for a particular $\mathfrak{g}, Y$.

4) $Y = Op_{{}^LG}(D^\times)$ – ${}^LG$ opers on $D^\times$

$$= \left\{ \partial_t + \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \Big/ {}^LN((t)) \qquad {}^LN = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$\begin{tikzcd} Op_{{}^LG}(D^\times) \arrow[d, "\alpha"', "\text{forgetful map}"] \\ Loc_{{}^LG}(D^\times) \end{tikzcd}$$

<u>Main conjecture</u> The corr. category over $Op_{{}^LG}(D^\times)$ is the category of $\hat{\mathfrak{g}}$-modules of critical level.

$$\begin{tikzcd} \hat{\mathfrak{g}}\text{-mod} \arrow[r] \arrow[d, "G((t)) \circlearrowleft"'] \arrow[dr, phantom, "\square"] & \mathcal{C}_{univ} \arrow[d, "G((t)) \circlearrowleft"] \\ Op_{{}^LG}(D^\times) \arrow[r, "\alpha"'] & Loc_{{}^LG}(D^\times) \end{tikzcd}$$

<u>Thm</u> (Feigin-F.) The center of the completed env. alg of $\hat{\mathfrak{g}}$ at critical level $\simeq \mathrm{Fun}(\mathrm{Op}_{{}^LG}(D^\times))$

$\circlearrowright\ \tilde{U} \supset Z$

This means that $\hat{\mathfrak{g}}_{crit}\text{-mod} \supset \hat{\mathfrak{g}}_{crit}\text{-mod}_\chi$ max. ideal

$$\begin{tikzcd} \hat{\mathfrak{g}}_{crit}\text{-mod} \arrow[d] & \supset \hat{\mathfrak{g}}_{crit}\text{-mod}_\chi \\ \mathrm{Op}_{{}^LG}(D^\times) \ni \chi \arrow[ur] & \end{tikzcd}$$

$\leftarrow$ those modules on which Z acts via χ $(m_\chi \mapsto 0)$.

$G((t)) \circlearrowright \hat{\mathfrak{g}}$
preserves central char

<u>conj</u>

$$\chi_1, \chi_2 \in \mathrm{Op}_{{}^LG}(D^\times) \xmapsto{\alpha} \sigma$$

$$\begin{array}{ccc} \hat{\mathfrak{g}}_{crit}\text{-mod}_{\chi_1} & \simeq & \hat{\mathfrak{g}}_{crit}\text{-mod}_{\chi_2} \\ \circlearrowleft & & \circlearrowleft \\ G((t)) & & G((t)) \end{array}$$

5) What is known? consider categories of H-C modules for various "compact subgroups" of $G((t))$

- unramified case $K = G[[t]]$

$$\hat{\mathfrak{g}}_{crit}\text{-mod}_\chi^{G[[t]]} \simeq \mathrm{Vect} \longleftrightarrow \Pi^{G(\mathcal{O})} \neq 0$$ iff σ-unramified and then it is one-dim

iff χ is unramified as a local system, $\chi \in \alpha^{-1}(\sigma_0)$

- Tamely ramified case: $\begin{array}{ccc} K = I & \longrightarrow & B \\ \cap & & \cap \\ G[[t]] & \xrightarrow{t=0} & G. \end{array}$

$$\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-}\mathrm{mod}^{I}_{\chi} \simeq D^b(\mathrm{QCoh}(\mathrm{Sp}_{\mathrm{monodromy}})) \longleftrightarrow$$ Kazhdan Lusztig and Ginzburg.

iff χ is tamely ramified