$\underline{g}$ simple Lie-alg

P^+ dominant weights

R^+ set of positive roots

Abelian ideals

$\Psi \subseteq R^+$ $\quad \alpha, \beta \in \Psi \Rightarrow \alpha+\beta \notin R^+$

$\alpha \in \Psi, \ \beta \in R^+, \ \alpha+\beta \in R^+ \Rightarrow \alpha+\beta \in \Psi$

Fix I to be index of simple roots

$\alpha \in R^+, \quad \alpha = \sum_{i \in I} d_i(\alpha)\alpha_i, \quad \theta \in R^+$ highest root

$\Psi_i = \{ \alpha \in R^+ : d_i(\alpha) = d_i(\theta) \} \quad i \in I$

is an abelian ideal in R^+

Ψ abelian. order $\leq_\Psi$ on P^+

$\lambda \leq_\Psi \mu \Leftrightarrow \mu - \lambda \in \mathbb{Z}_+$ - span of Ψ

$\{ \mu \in P^+ : \mu \leq_\Psi \lambda \}$ is a finite set.

$(P^+, \leq_\Psi) \quad \mu \leq_\Psi \lambda,$

$ht_\Psi(\lambda - \mu) = \min \sum_{\psi \in \Psi} n_\psi \quad \left(\lambda - \mu = \sum n_\psi \psi\right)$

$d_\Psi(\lambda, \mu) = ht_\Psi(\lambda - \mu) \quad \Psi = \Psi_i$

$\Rightarrow d_\Psi(\lambda, \mu) = d_\Psi(\lambda, \nu) + d_\Psi(\nu, \mu)$ if $\mu \leq_\Psi \nu \leq_\Psi \lambda$

(distance function). Poset is graded.

Subset $F \subseteq P^+$ is interval closed if $\lambda, \mu \in F, \ \lambda \leq_\Psi \mu$

$\Rightarrow \nu \in F$ if $\lambda \leq_\Psi \nu \leq_\Psi \mu$.

Algebras $F \subseteq P^+, \ \Psi = \Psi_i \quad i \in I$.

$$A_\Psi(F) = \bigoplus_{\substack{\lambda, \mu \in F \\ \lambda \leq_\Psi \mu}} \mathrm{Hom}_{\underline{g}}\left(V(\mu), S^{d(\mu,\lambda)}(\underline{g}) \otimes V(\lambda)\right)$$

where $V(\mu)$ is irrep of $\underline{g}$ with highest wt. μ

$S(\underline{g})$ is symmetric algebra of $\underline{g}$

Multiplication

$$V(\mu) \xrightarrow{f} S^{d(\lambda,\mu)}(\mathfrak{g}) \otimes V(\lambda) \longrightarrow S^{d(\lambda,\mu)}(\mathfrak{g}) \otimes S^{d(\lambda,\lambda')}(\mathfrak{g}) \otimes V(\lambda')$$

$$V(\nu) \xrightarrow{f'} S^{d(\nu,\lambda')}(\mathfrak{g}) \otimes V(\lambda')$$

$f \cdot f' = 0$, $f\lambda \neq \nu$, $\lambda = \nu$,

$$\downarrow$$

$$S^{d(\lambda,\mu)+d(\lambda,\lambda')}(\mathfrak{g}) \otimes V(\lambda')$$

$$\parallel$$

$$S^{d(\mu,\lambda')}(\mathfrak{g}) \otimes V(\lambda') \ .$$

$A_\psi(F)$ is a $\mathbb{Z}^+$-graded assoc. alg.

$$A_\psi(F)[k] = \bigoplus_{\substack{\lambda \leq_\psi \mu \\ d(\lambda,\mu)=k}} \mathrm{Hom}_{\mathfrak{g}}\left(V(\mu),\ S^{d(\mu,\lambda)}(\mathfrak{g}) \otimes V(\lambda)\right)$$

$A_\psi(P^+) = A_\psi$

For $\lambda \in P^+$, $A_\psi(\lambda) = A_\psi(\{\mu \in P^+ : \mu \leq_\psi \lambda\})$

$\uparrow$ fin-dim. $\mathbb{Z}_+$-graded assoc. alg.

Thm (C, Greenstein)

$\psi = \psi_i$ $\quad i \in I$.

(i) let $F \subseteq P^+$ be an interval closed set. Then the grading on $A_\psi(F)$ is Koszul.

(ii) A_ψ = direct limit of $A_\psi(F)$, F interval-closed. A_ψ is Koszul.

(iii) global dimension of A_ψ, $A_\psi(F)$ is $|\psi|$.

(iv) quadratic dual is $A_\psi(F)$, obtained by replacing $S(\mathfrak{g})$ by $\Lambda(\mathfrak{g})$.

Connection with KR-modules

$\mathfrak{g} \otimes \mathbb{C}[t]$ is $\mathbb{Z}_+$-graded Lie alg.

$\mathcal{G}$ category of $\mathbb{Z}_+$-graded $\mathfrak{g} \otimes \mathbb{C}[t]$-modules.

(v) category of right modules for $A_\Psi(\lambda)$ is equivalent to a full subcategory of $\mathcal{G}$.

(vi) $\lambda = m\omega_i$, $i \in I$, $d_i(\theta) = 2$.

Then right-projective modules for $A_\Psi(\lambda)$ corresp to $KR(m\omega_i)$.

In general, right proj modules corresp to the projective cover of min affinization of λ.

One knows character formulas for the projectives.
($\operatorname{supp} \lambda = \{ i : \lambda(\alpha_i^\vee) \neq 0 \} \subseteq$ conn. subdiag of type A)

Motivations

Category of f.d. reps of $U_q(L\mathfrak{g}) \leftrightarrow U_q(\mathfrak{g})$

$V_q = \bigoplus m_\mu(V_q)\, V(\mu)$.

$q \to 1$ Category of f.d. reps of $U(L\mathfrak{g}) \leftrightarrow U(\mathfrak{g})$.

$$\mathfrak{g} \otimes \frac{\mathbb{C}[t, t^{-1}]}{(f)} \simeq \mathfrak{g} \otimes \frac{\mathbb{C}[t]}{(f)} = \bigoplus \mathfrak{g} \otimes \frac{\mathbb{C}[t]}{(t-a)^{r_i}}$$

assume $a_i = 0$. looking at reps of $\mathfrak{g} \otimes \frac{\mathbb{C}[t]}{t^N \mathbb{C}[t]}$.

$N=1$ $\quad \mathfrak{g}$

$N=2$ $\quad \mathfrak{g} \otimes \frac{\mathbb{C}[t]}{t^2\mathbb{C}[t]}$.

One knows [C] that specializations of KR_q (mω_i) are modules for this algebra.

$$\mathfrak{g} \otimes \frac{\mathbb{C}[t]}{(t^2)} \simeq \underset{\text{grade } 0}{\mathfrak{g}} \ltimes \underset{\text{grade } 1}{\mathfrak{g}_{ad}}$$ are $\mathbb{Z}_+$-graded.

$U(\mathfrak{g} \ltimes \mathfrak{g}_{ad})$ is $\mathbb{Z}_+$-graded.

$$U(\mathfrak{g} \ltimes \mathfrak{g}_{ad}) \simeq U(\mathfrak{g}_{ad}) \otimes U(\mathfrak{g})$$
$$\simeq S(\mathfrak{g}_{ad}) \otimes U(\mathfrak{g})$$
$$U(\mathfrak{g} \ltimes \mathfrak{g}_{ad})[k] = S^k(\mathfrak{g}) \otimes U(\mathfrak{g})$$

$\mathcal{G}_2$ = category of $\mathfrak{g} \ltimes \mathfrak{g}_{ad}$-modules, graded by $\mathbb{Z}_+$.

$V = \bigoplus_{k \geq 0} V[k]$, $\dim V[k] < \infty$

$\mathfrak{g}\, V[k] \subseteq V[k]$

$\mathfrak{g}_{ad}\, V[k] \subseteq V[k+1]$

$V \in \operatorname{Ob} \mathcal{G}_2$

$$\operatorname{ch}_{\mathfrak{g}}(V) = \sum_{\substack{\mu \in P^+ \\ k \in \mathbb{Z}_+}} (\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), V[k]))\, t^k.$$

Simple object: $V \in \mathcal{G}_2$, $V = V[k]$

$V[k]$ irred $\mathfrak{g}$-module

irred obj $\Leftrightarrow$ (λ, r) $\lambda \in P^+$, $r \in \mathbb{Z}_+$.

$\mathcal{G}_2$ is not semisimple. $\mathfrak{g} \ltimes \mathfrak{g}_{ad} \in \mathcal{G}_2$.

Projective objects in $\mathcal{G}_2$.

$V \in \mathcal{G}$, $V = V[k]$

as $\mathfrak{g} \ltimes \mathfrak{g}_{ad}$-modules

$$P(V) = U(\mathfrak{g} \ltimes \mathfrak{g}_{ad}) \otimes_{U(\mathfrak{g})} V \simeq S(\mathfrak{g}_{ad}) \otimes V$$

$$(\lambda, r) \longrightarrow V(\lambda, r)$$

$$P(V(\lambda, r)) \longrightarrow V(\lambda, r) \longrightarrow 0$$

$$1 \otimes v \longmapsto v$$

$$x \otimes v \longmapsto 0, \quad x \in \mathfrak{g}_{ad}$$

$\mathcal{G}_2(\lambda,\psi)$ is equivalent to category of right module for $A_\lambda(\psi)$.

$P(\mu, d(\lambda,\mu))^{(\lambda,\psi)}$ = KR-module.

If $\mu = \lambda = m\omega_i$ then this is the KR-module.
and one has a comp. rep. theoretic way of describing chars.

$$\mathcal{G}_2 \longrightarrow \mathcal{G}_2(\lambda,\psi)$$

$$\mathrm{Ext}^i_{\mathcal{G}_2} \simeq \mathrm{Ext}^i_{\mathcal{G}_2(\lambda,\psi)}(V(\lambda,r), V(\mu,s)) = 0 \qquad i \neq s-r$$