13 Mai 2008

V. Kac: Integrable systems and related algebraic structures

(1) Four fundamental theories:

$$\begin{tikzcd} \mathrm{Cl\,FT} & \mathrm{QFT} \arrow[l] \arrow[d] \\ \mathrm{Cl\,M} \arrow[from=u] & \mathrm{QM} \arrow[l] \end{tikzcd}$$

corresponding algebraic structures

$$\begin{tikzcd} \text{Poisson vertex algebras} \arrow[d] & \text{vertex algebras} \arrow[l] \\ \text{poisson algebras} & \text{associative algebras} \arrow[l] \end{tikzcd}$$

(2) what is an integrable system:

Finite dimensional case.

Evolution equation ODE:

$$(1) \quad \frac{du}{dt} = f(u)$$

$$u = u(x,t) = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}, \quad f(u) = \begin{pmatrix} f_1(u) \\ \vdots \\ f_d(u) \end{pmatrix}$$

Phase space : coordinates $u_1, \dots, u_d$,
where $u_i(\cdot)$ are functions on a manifold

Integral of motion

function $\varphi = \varphi(u)$ on phase space.
s.t $\frac{d\varphi}{dt} = 0$ by (1).

By chain rule $\Rightarrow$
$X_f \varphi = 0$ where
$X_f = \sum_i f_i \frac{\partial}{\partial u_i}$

3 Infinite dimensional case

Evolution eqn. PDE

(6) $\frac{du}{dt} = f(u, u', u'', \dots)$

Phase space infinite dimensional:
space of functions is
$\hat{\mathcal{F}} = \mathbb{C}[u_i^{(n)} \mid i = 1, \dots, d, n \in \mathbb{Z}_+\}$

where $u_i + u_i(x,t)$ are smooth functions in one indeterminate x

$u_i' = \frac{d}{dx} u_i$; t parameter

$\partial = \frac{d}{dx}$: $u^{(n)} \mapsto u^{(n+1)}$, a derivation of $\tilde{\mathcal{F}}$

integral of motion is local functional

$\int \varphi$ = image of φ in $\tilde{\mathcal{F}} / \partial \tilde{\mathcal{F}}$
(univ. spc where integration by parts holds)

special case of ev. eqn:

Hamiltonian eqn

(2) $\frac{du}{dt} = \{h, u\}$ where $h \in \mathcal{F}(u_1, \dots, u_d)$
and $\{\cdot,\cdot\}$ is a Poisson bracket
making $\tilde{\mathcal{F}}$ a poisson algebra

a) product unital commutative associative

b) $\{\cdot,\cdot\}$ is a Lie algebra

c) Leibniz rule : $\{a, bc\} = \{a,b\}c + b\{a,c\}$

Remark

To construct a Poisson bracket it suffices to define $\{u_i, u_j\} = -\{u_j, u_i\}$ on each pair and check Jacobi identity

what is Hamiltonian PDE?

Need to define an approximate "Poisson bracket"

Physicists write

(7) $\{u_i(x), u_j(y)\} = H_{ji}(u(y), u'(y), \ldots, \partial_y)\,\delta(x-y)$

where the δ-function is given by

—

Extend (7) by Leibniz rule

$$\{f(x), g(y)\}$$

$$= \sum_{\substack{m,n \in \mathbb{Z}_+ \\ i,j}} \frac{\partial f}{\partial u_i^{(m)}} \frac{\partial g}{\partial u_j^{(n)}} \partial_x^m \partial_y^n \{u_i(x), u_j(y)\}.$$

The bracket should be skew-comm. and satisfy Jacobi id.

(10) $\{f_\lambda g\}$

$$= \sum_{\substack{m,n\in\mathbb{Z}_+ \\ i,j}} \frac{\partial g}{\partial u_j^{(n)}} (\partial+\lambda)^n \{u_{j\,\partial+\lambda}\, u_i\}_\rightarrow (-\partial-\lambda)^m \frac{\partial f}{\partial u_i^{(m)}}$$

$$\{u_{j\,\partial+\lambda}\, u_i\}_\rightarrow = H_{ij}(\partial+\lambda).$$

we obtain the structure of a Poisson vertex algebra.

Axioms of a Lie conformal algebra.

1). $\{P(\partial)a_\lambda b\} = P(-\lambda)\{a_\lambda b\}$; $\{a_\lambda P(\partial) b\} = P(\partial+\lambda)\{a_\lambda b\}$

2). $\{b_\lambda a\} = -\{a_{-\lambda-\partial} b\}$.

3) $\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\}$.

Thm (D'Andrea VK 1997). All simple finite Lie conformal algebras are:

1). $\mathbb{C}[\partial]\mathfrak{g}$, $[a_\lambda b] = [a, b]$ where $a, b \in \mathfrak{g}$ simple f.d Lie alg.

2) $\mathbb{C}[\partial]a$, $[a_\lambda a] = (\partial+2\lambda)a$

<u>Remark</u> Generalized GFZ bracket and Virasoro-Magri bracket are central extensions of those;

1) $[a_\lambda b] = [ab] + \lambda(a,b)K$. $K \in \mathbb{C}$, $a,b \in \mathfrak{g}$

2). $[a_\lambda a] = (\partial + 2\lambda)a + c\lambda^3$, $c \in \mathbb{C}$

<u>Key observation</u>

a) If V is Lie conformal alg. then $V/\partial V$ has a well defined Lie alg. bracket

(11) $\{\int f, \int g\} = \int \{f_\lambda g\}|_{\lambda=0}$

V being a $V/\partial V$-module via

(12) $(\int f)g = \{f_\lambda g\}|_{\lambda=0}$.

If, in addition V is a PVA, then

$$\int f \longmapsto \{\int f_\lambda \cdot\}|_{\lambda=0}$$

defines a Lie-alg homomorphism $V/\partial V$ to evolutionary vector fields.

By Leibniz rule (2) implies that
$\frac{df}{dt} = \{h, f\}$ for any $f \in \tilde{\mathcal{F}}$ of (2)

$\Rightarrow$ f is an integral of motion iff

(4) $\{h, f\} = 0$

By Jacobi id.
$f \mapsto X^f \cdot = \{f, \cdot\}$ is a Lie alg hom.
of $(\mathcal{F}, \{\cdot,\cdot\})$ to Lie alg. of vector fields on phase space.

A Hamiltonian eq is called integrable if h is contained in an abelian Poisson subalgebra of $(\mathcal{F}, \{\})$ of maximal possible G/K dimension. Then the Liouville thm. guarantees integrability.

All Poisson brackets linear in each u
$\Leftrightarrow$ all d-dimensional $\mathfrak{g}$

e.g.

$\mathcal{F}$ = functions in $p_i = m_i v_i$

and $q_i = x_i$

Poisson bracket

$\{p_i\, p_j\} = 0 = \{q_i\, q_j\}$,

$\{p_i\, q_j\} = -\{q_j, p_i\} = \delta_{ij}$

$h = \frac{1}{2} \sum m_i v_i^2 + V(q)$,

Hamiltonian eqn $\frac{du}{dt} = \{h, u\}$

$u = \begin{pmatrix} p_1 \\ \vdots \\ q_1 \\ \vdots \end{pmatrix} \iff \frac{dp_i}{dt} = -\frac{\partial V}{\partial q}, \quad m_i \frac{dq_i}{dt} = p_i$

Endow $\tilde{\mathcal{F}}$ w/ a λ-bracket $\{\cdot{}_\lambda\cdot\}$ making it a PVA

Hamiltonian eqn

(B) $$\frac{du}{dt} = \{\textstyle\int h_\lambda u\}|_{\lambda=0}, \quad \textstyle\int h \in \tilde{\mathcal{F}}/\partial\tilde{\mathcal{F}}$$

$\int f$ is an ~~x~~ in $\tilde{\mathcal{F}}$ integral of motion of (B) if ~~it~~ it is ~~or~~ in involution w/ $\int h$:

$$\{\textstyle\int h, \int f\} = 0 \quad \text{in } \tilde{\mathcal{F}}/\partial\tilde{\mathcal{F}}$$

Integrals of motion in involution produce heirarchy of Hamiltonian eqns'

More familiar form

$$\frac{du}{dt} = H(\partial)\frac{\delta h}{\delta u},$$

$$\frac{\delta h}{\delta u_i} = \sum(-\partial)^n\frac{\partial h}{\partial u_i^{(n)}}.$$

is the variational derivative

$$\{\textstyle\int h, \int f\} = \int\left(\frac{\delta h}{\delta u}\right)^T H(\partial)\frac{\delta f}{\delta u}$$

is bracket on $\tilde{\mathcal{F}}/\partial\tilde{\mathcal{F}}$

A Hamiltonian hierarchy is integrable if all $\int h$ is contained in an infinite dim abelian subspace of the Lie algebra $\tilde{\mathcal{F}}/\partial\tilde{\mathcal{F}}$

Problems:

a) classify all PVA

b) For PVA classify all max abelian ∞-dim sub Lie algebras of $\tilde{\mathcal{F}}/\partial\tilde{\mathcal{F}}$

b). simplest example (GFZ).

$\tilde{\mathcal{F}} = \mathbb{C}[v, v', v'', \ldots]$, $\{v_\lambda, v\} = \lambda$.

$\tilde{F}/\partial\tilde{F}$ has bracket

$$\{\int f, \int g\} = \int \frac{\delta g}{\delta u} \partial \frac{\delta f}{\delta u}$$

Examples of max abelian subalgs:

$A_0 = \{1, v, v^2, v^3, \ldots\}$

($\dot{v} = \{h_\lambda v\}|_{\lambda=0}$ is the dispersionless

KdV hierarchy $\frac{dv}{dt_n} = v' v^{n-1}$, $n \geq 1$, $\frac{du}{dt_{1,0}} = 0$

$A_\infty = \{1, v, v^2, v'^2, v''^2, \ldots\}$

corresponding hierarchy is linear

$(c \neq 0)$ $A_c = \{1, v, v^2, \frac{1}{2}v^3 + \frac{1}{2}cvv'', \frac{5}{8}v^4 + \frac{5}{3}cv^2v''$
$+ \frac{5}{6}cvv'^2 + \frac{1}{2}c^2vv'''', \ldots\}$

corresponds to KdV hierarchy

The first few eqns of KdV hierarchy are.

$$\frac{\delta v}{\delta t_n} = \partial\left(\frac{\delta h_n}{\delta u}\right)$$

$$\frac{du}{dt_{-1}} = 0, \quad \frac{du}{dt_0} = 0, \quad \frac{du}{dt_1} = v'$$

$$\frac{du}{dt_2} = 3vv' + cv''' \quad \text{KdV eqn}$$

etc.

How is A_c constructed?

<u>conjecture</u> The A_c ~~are all~~ classify (b).

A difficulty in classification of integrable eqns is that repetitions may occur.

$$\frac{du}{dt} = \{\textstyle\int h_{0\lambda}\, u\}_1 \Big|_{\lambda=0} = \{\textstyle\int h_{1\lambda}\, u\}_0 \Big|_{\lambda=0}$$

for 2 - different λ - brackets.

eg $\{u_\lambda u\}_0 = \lambda$ GFZ and

$\{u_\lambda u\}_1 = (\partial + 2\lambda) u + c\lambda^3$ Virasoro - Magri

are compatible.

Lenard scheme: suppose $\tilde{F}$ is endowed w/ two compatible PVA λ brackets $\{\cdot_\lambda \cdot\}_i$ $i = 0, 1$

and $\exists$ $\int h_i \in \tilde{F} / \partial \tilde{F}$ s.t.

$$\{\textstyle\int h_{0\lambda}\, u\}_1 \Big|_{\lambda=0} = \{\textstyle\int h_{1\lambda}\, u\}_0 \Big|_{\lambda=0}.$$

then under mild assumptions $\exists$ sequence $\int h_n$ $n \in \mathbb{Z}_+$ w/ $\int h_0, \int h_1$ s.t.

$$\{\textstyle\int h_{n-1\,\lambda}\, u\}_1 \Big|_{\lambda=0} = \{\textstyle\int h_{n\lambda}\, u\}_0 \Big|_{\lambda=0}.$$

span $\int h_n$ is ∞-dim abelian subspace of both $\{\cdot_\lambda \cdot\}$

<u>Example</u> For GFZ - Vir - Magri.
pair of λ brackets take.

$\int h_0 = \int u, \quad \int h_1 = \frac{1}{2}\int u^2.$

Then (16) holds and we get KdV ~~eqs~~.
sequence $\int h_n$.

What about problem (a)?
classify simple P. algebras.
In dim < ∞ the answer is not interesting

$\{p_i, q_j\} = \delta_{ij}$ on $\mathbb{C}[[p,q]]$.
are all of them

<u>Conjecture 2</u> Let V be a simple PVA
generated by L st $\{L_\lambda L\} = (\partial + 2\lambda)L + c\lambda^3$
and finitely many ~~et~~ s-primary elements.

$\{L_\lambda W_j\} = (\partial + \Delta_j \lambda) W_j$

$\Delta_j \in \mathbb{Z}_+ + \frac{1}{2}.$

Then $V \cong$ to one of the classical W algebras $W^c(\mathfrak{g}, f)$ ← nilpotent
↑
reductive

Proved for $s=1$.

Example

$W^c(sl_3, f_{principal})$

$= \mathbb{C}[L, W, L', W', \dots]$

$\{L_\lambda L\} = (\partial + 2\lambda)L + \frac{\lambda^3}{12}c.$

$\{L_\lambda W\} = (\partial + 3\lambda)W$

$\{W_\lambda W\} = (\partial + 2\lambda)\left(L^2 + \frac{1}{128}cL''\right) + (\partial + 2\lambda)^3 \frac{5}{384}cL$

$+ \frac{c^3\lambda^5}{11.32}.$

$W^c(sp_4, f_{principal})$

$W^c(G_2, f_{principal})$

$W^c(sl_2, f) =$ Virasoro - Magri.

data $\mathfrak{g}$ - simple fd Lie (super) algebra.

f - nilpotent

include $\mathfrak{g} \supset sl_2 = \langle e, h, f \rangle$ w/ respect to $\operatorname{ad} h$

$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ let $M = \mathfrak{g}_{\geq 1/2}$, $\supset \mathfrak{m} = \mathfrak{g}_{\geq 1}$.

classical affine W-alg
- generalized DS reduction
KdV-type eqns

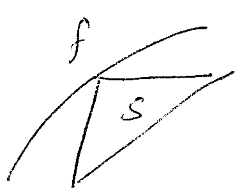
Affine W-algebra
rep theory of Virasoro like algebras

$$\begin{array}{ccc}
W^k(\mathfrak{g}, f) & \longleftarrow & W^k(\mathfrak{g}, f) \\
 & & \downarrow \\
W^{fin}(\mathfrak{g}, f) & \longleftarrow & W^{fin}(\mathfrak{g}, f) \\
\| & & \| \\
(S(\mathfrak{g})/S(\mathfrak{g})I)^M & & U(\mathfrak{g})/U(\mathfrak{g})I)^M
\end{array}$$

Slodowy slice

quantization of Slodowy slice
controls primitive ideals of $\mathfrak{g}$

$S = f + \ker e \subset \mathfrak{g}$
$G \cdot f$.



for subregular f:

$S \longrightarrow \mathfrak{g}/G = \mathfrak{h}/W$

is semiuniversal deformation of simple rational singularity.

~~Yes~~ True for all f? Yes if we allow only Poisson deformations.