James' Conjecture for Hecke Algebras of Exceptional Type

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"Everybody" knows that the irreducible representations of the symmetric group $\mathfrak{S}_n$ are parametrised by the partitions of n

(... if we are working over a field of characteristic 0).

- JAMES 1970's: "characteristic-free" approach
  (Specht modules S^λ, simple modules $D^\lambda = S^\lambda/\text{rad}(S^\lambda)$, ...)
- DIPPER AND JAMES 1980's: q-version for Hecke algebra of $\mathfrak{S}_n$
  (Specht modules S_q^λ, simple modules $D_q^\lambda = S_q^\lambda/\text{rad}(S_q^\lambda)$, ...)

$$\boxed{\dim D_q^\lambda \quad = \quad ?}$$

- JAMES' CONJECTURE 1990: reduction to characteristic 0 and an explicitly given finite number of "small" characteristic cases

Aim of this talk: $\mathfrak{S}_n \cong W(A_{n-1}) \quad \rightsquigarrow \quad$ any type of W

W finite Weyl group with generating set S, $R \subseteq \mathbb{C}$ any subring,
$\mathcal{H} = \mathcal{H}_A(W)$ generic Iwahori–Hecke algebra over $A = R[u^{1/2}, u^{-1/2}]$.

- free A-module, basis $\{T_w \mid w \in W\}$;
- associative multiplication

  $T_w = T_{s_1} \cdots T_{s_l}$ if $w = s_1 \cdots s_l$ ($s_i \in S$) minimal expression;

  $T_s^2 = uT_1 + (u-1)T_s$ for $s \in S$.

Specialisation: ring homomorphism $A \to k$ (k field), $u \mapsto \xi \in k$.

$\mathcal{H}_{k,\xi} = k \otimes_A \mathcal{H}$ associative algebra over k; basis $\{T_w \mid w \in W\}$.

$$T_s^2 = \xi T_1 + (\xi - 1) T_s \text{ for } s \in S.$$

$\xi = 1 \rightsquigarrow T_s^2 = T_1$ for all $s \in S$ and so $\mathcal{H}_{k,1} = k[W]$ (group algebra).

Problem: Determine $\mathrm{Irr}(\mathcal{H}_{k,\xi})$ for various k, ξ.

A general approach is given by the theory of "cellular algebras".

Let H be any associative algebra over a commutative ring A.

Definition. Graham and Lehrer (1996)

A "cell datum" $(\Lambda, M, C, *)$ for H has to satisfy:

- Λ is a partially ordered set (with partial ordered denoted by $\trianglelefteq$), $\{M(\lambda) \mid \lambda \in \Lambda\}$ are finite sets and

  $$\{C^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s},\mathfrak{t} \in M(\lambda)\} \quad \text{is an } A\text{-basis of } H.$$

- $*\colon H \to H$ is an A-linear anti-involution such that

  $$(C^\lambda_{\mathfrak{s},\mathfrak{t}})^* = C^\lambda_{\mathfrak{t},\mathfrak{s}} \text{ for all } \lambda \in \Lambda \text{ and } \mathfrak{s},\mathfrak{t} \in M(\lambda).$$

- For $h \in H$: $\quad hC^\lambda_{\mathfrak{s},\mathfrak{t}} = \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s})\, C^\lambda_{\mathfrak{s}',\mathfrak{t}} + \text{"lower terms"}$,
  - where "lower terms" means: combination of $C^\mu_{\mathfrak{s}'',\mathfrak{t}''}$ where $\mu \triangleleft \lambda$,
  - and $r_h(\mathfrak{s}', \mathfrak{s}) \in A$ is independent of $\mathfrak{t}$.

Fix $\lambda \in \Lambda$. Let Δ^λ be a free A-module with basis $\{e_{\mathfrak{s}} \mid \mathfrak{s} \in M(\lambda)\}$.

Left action of H: $$h.e_{\mathfrak{s}} = \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s})\, e_{\mathfrak{s}'}.$$

H-equivariant bilinear form $\phi^\lambda \colon \Delta^\lambda \times \Delta^\lambda \to A$, defined by:

$$C^\lambda_{\mathfrak{s}',\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{t}'} = \phi^\lambda(e_{\mathfrak{s}}, e_{\mathfrak{t}})\, C^\lambda_{\mathfrak{s}',\mathfrak{t}'} \quad + \quad \text{"lower terms"}.$$

Specialisation $A \to k$ (k field) $\rightsquigarrow$ k-algebra $H_k = k \otimes_A H$ and cell modules $\Delta^\lambda_k = k \otimes_A \Delta^\lambda$, with induced bilinear forms ϕ^λ_k.

Theorem. Graham–Lehrer (1996)

Assume $\dim H_k < \infty$. Set $L^\lambda_k = \Delta^\lambda_k / \operatorname{rad}(\phi^\lambda_k)$ for $\lambda \in \Lambda$. Then

$$\operatorname{Irr}(H_k) = \{L^\lambda_k \mid \lambda \in \Lambda^\circ_k\} \quad \text{where} \quad \Lambda^\circ_k = \{\lambda \in \Lambda \mid \phi^\lambda_k \neq 0\}.$$

H_k is semisimple if and only if $\phi^\lambda_k \neq 0$ and $L^\lambda_k = \Delta^\lambda_k$ for all $\lambda \in \Lambda$.

Example. Let $H = \mathcal{H}_A(W)$ where $W = W(A_{n-1}) \cong \mathfrak{S}_n$.

- Cellular basis constructed by Murphy (1992) by a purely combinatorial argument (before "cell data" were introduced).
  - $\Lambda =$ set of all partitions of n,
  - $\trianglelefteq =$ dominance order,
  - $M(\lambda) =$ set of standard λ-tableaux,
  - anti-involution $T_w^* = T_{w^{-1}}$ $(w \in W)$,
  - $C_{\mathfrak{s},\mathfrak{t}}^\lambda =$ explicit formula in terms of basis elements T_w.
- The modules Δ^λ are the Dipper–James Specht modules S^λ.

Dipper–James–Murphy (1995): Generalisation to type B_n where $\Lambda =$ set of pairs of partitions of total size n, ...

How to define a "cell datum", for example, in type E_8 ?

Aim: General construction of a "cell datum" for $\mathcal{H} = \mathcal{H}_A(W)$, where W is any finite Weyl group. Start with:

$$\text{Kazhdan–Lusztig basis } \{\mathbf{C}_w \mid w \in W\} \text{ of } \mathcal{H}.$$

This gives rise to partitions

$$W = \coprod \{ \text{ left cells } \} = \coprod \{ \text{ right cells } \} = \coprod \{ \text{ two-sided cells } \}$$

and partial orderings $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$ on left, right, two-sided cells.

Furthermore, there is a natural map

$$\text{Irr}(W) = \{E^\lambda \mid \lambda \in \Lambda\} \to \{ \text{ two-sided cells } \}, \qquad E^\lambda \mapsto \mathfrak{C}_\lambda.$$

(Indeed, each left cell gives rise to a left module of $\mathcal{H}$, and of $\mathbb{Q}[W]$. Given λ, let Γ be a left cell such that E^λ appears in the left $\mathbb{Q}[W]$-module carried by Γ. All such Γ lie in the same two-sided cell; this defines $\mathfrak{C}_\lambda$.)

Using the map $E^\lambda \mapsto \mathfrak{C}_\lambda$, we define a partial order on Λ:

$$\lambda \trianglelefteq \mu \quad \overset{\text{def}}{\Leftrightarrow} \quad \lambda = \mu \quad \text{or} \quad \mathfrak{C}_\lambda \quad \lneqq_{\mathcal{LR}} \quad \mathfrak{C}_\mu.$$

Ingredients for a cell datum of $\mathcal{H}$:

- $(\Lambda, \trianglelefteq)$ as above, $\operatorname{Irr}(W) = \{E^\lambda \mid \lambda \in \Lambda\}$; $M(\lambda) =$ basis of E^λ;
- anti-involution $T_w^* = T_{w^{-1}}$ $(w \in W)$; also have $\mathbf{C}_w^* = \mathbf{C}_{w^{-1}}$.

Theorem. G. (2006)

Assume that $R \subseteq \mathbb{C}$ is a subring such that "bad primes" for W are invertible in R. Then $\mathcal{H}$ admits a "cell datum" such that

$$C^\lambda_{\mathfrak{s},\mathfrak{t}} \text{ is an integral linear combination of } \mathbf{C}_w\text{'s where } w \in \mathfrak{C}_\lambda.$$

A_n:	no bad p;	$G_2,\ F_4,\ E_6,\ E_7$:	$p \neq 2, 3$;
$B_n,\ C_n,\ D_n$:	$p \neq 2$;	E_8:	$p \neq 2, 3, 5$.

(Proof uses Lusztig's homomorphism $\Phi\colon \mathcal{H} \to A \otimes_{\mathbb{Z}} \mathbf{J}$ where $\mathbf{J}$ "asymptotic ring".)

Example. Let $W = W(A_{n-1}) \cong \mathfrak{S}_n$. Kazhdan–Lusztig (1979):

The $\mathbb{Q}[W]$-modules carried by the left cells are all irreducible.

Expression for $C^{\lambda}_{\mathfrak{s},\mathfrak{t}}$ reduces to one term. Hence

$$\{\mathbf{C}_w \mid w \in W\} \text{ is a cellular basis;}$$

$$\trianglelefteq \text{ (induced by } \leqslant_{\mathcal{LR}}) \quad \leftrightarrow \quad \text{dominance order on partitions.}$$

This example already appeared in the article of Graham and Lehrer.

- McDonough and Pallikaros (2005):
  Cell modules Δ^{λ} (=Kazhdan–Lusztig cell modules in this case) are canonically isomorphic to Dipper–James Specht modules.
- G. (2006): Describes base change between the Kazhdan–Lusztig basis and the Murphy basis of $\mathcal{H}$.

Let $W = W(B_2)$, with generators $S = \{s_1, s_2\}$ such that $(s_1 s_2)^4 = 1$.

$$\mathrm{Irr}_{\mathbb{Q}}(W) = \{\mathbf{1}, \varepsilon_1, \varepsilon_2, \varepsilon, r\}$$

where $\mathbf{1}$ =unit, ε =sign, $\dim \varepsilon_1 = \dim \varepsilon_2 = 1$, $\dim r = 2$. Bad: 2.

$M(r) = \{1, 2\}$; otherwise $M(\lambda) = \{1\}$; $\{\mathbf{1}\} \lhd \{\varepsilon_1, \varepsilon_2, r\} \lhd \{\varepsilon\}$.

$$\begin{aligned} C^{\mathbf{1}}_{1,1} &= \mathbf{C}_1, & C^{r}_{1,1} &= \mathbf{C}_{s_1} + \mathbf{C}_{s_1 s_2 s_1}, \\ C^{\varepsilon}_{1,1} &= \mathbf{C}_{s_1 s_2 s_1 s_2}, & C^{r}_{1,2} &= -2\mathbf{C}_{s_1 s_2}, \\ C^{\varepsilon_1}_{1,1} &= \mathbf{C}_{s_2} - \mathbf{C}_{s_2 s_1 s_2}, & C^{r}_{2,1} &= -2\mathbf{C}_{s_2 s_1}, \\ C^{\varepsilon_2}_{1,1} &= \mathbf{C}_{s_1} - \mathbf{C}_{s_1 s_2 s_1}, & C^{r}_{2,2} &= 2\mathbf{C}_{s_2} + 2\mathbf{C}_{s_2 s_1 s_2}. \end{aligned}$$

Remark: The Theorem also works if $\mathcal{H}$ is a 2-parameter algebra of type G_2 or F_4 (any choice of parameters, G. 2007) and in the "asymptotic case" of type B_n (Bonnafé–G.–Iancu 2003–2006). One needs to verify conjectures **P1–P15** in Lusztig's book (2003).

Assume from now on that all bad primes are invertible in $R \subseteq \mathbb{C}$.

Consider a specialisation $A \to k$ (k field), $u \mapsto \xi$. Write

$$\mathcal{H}_\xi = k \otimes_A \mathcal{H} \quad \text{and} \quad \Delta_\xi^\lambda = k \otimes_A \Delta^\lambda, \quad \text{with induced form } \phi_\xi^\lambda.$$

Set $L_\xi^\lambda = \Delta_\xi^\lambda / \mathrm{rad}(\phi_\xi^\lambda)$. Then

$$\mathrm{Irr}(\mathcal{H}_\xi) = \{L_\xi^\lambda \mid \lambda \in \Lambda_\xi^\circ\} \qquad \text{where} \qquad \Lambda_\xi^\circ = \{\lambda \in \Lambda \mid \phi_\xi^\lambda \neq 0\}.$$

$\mathcal{H}_\xi$ is semisimple if and only if $\Lambda_\xi^\circ = \Lambda$ and $L_\xi^\lambda = \Delta_\xi^\lambda$ for all $\lambda \in \Lambda$.

Major open problem:

Find explicit formulas for $\dim L_\xi^\lambda$ or, equivalently, for $\dim \mathrm{rad}(\phi_\xi^\lambda)$.

In particular, determine the subset $\Lambda_\xi^\circ \subseteq \Lambda$.

Furthermore, determine the decomposition matrix

$$D_\xi = \big([\Delta_\xi^\lambda : L_\xi^\mu]\big)_{\lambda \in \Lambda,\, \mu \in \Lambda_\xi^\circ}$$

Let $k = \overline{\mathbb{F}}_\ell$. Then $\xi \in k^\times$ has finite order. Let

$$e = \min\{i \geqslant 2 \mid 1 + \xi + \xi^2 + \cdots + \xi^{i-1} = 0\}$$

and consider the $\mathbb{C}$-algebra $\mathcal{H}_{\zeta_e}$ where $\zeta_e = \sqrt[e]{1} \in \mathbb{C}$.

$$\operatorname{Irr}(\mathcal{H}_{\zeta_e}) = \{L^\lambda_{\zeta_e} \mid \lambda \in \Lambda^\circ_{\zeta_e}\} \qquad \text{where} \qquad \Lambda^\circ_{\zeta_e} = \{\lambda \in \Lambda \mid \phi^\lambda_{\zeta_e} \neq 0\}.$$

Proposition. (G. 1990) Assume that $\ell \gg 0$.

- $\Lambda^\circ_\xi = \Lambda^\circ_{\zeta_e}$ and $\dim L^\mu_\xi = \dim L^\mu_{\zeta_e}$ $\quad \forall \mu \in \Lambda^\circ_{\zeta_e}$.
- $[\Delta^\lambda_\xi : L^\mu_\xi] = [\Delta^\lambda_{\zeta_e} : L^\mu_{\zeta_e}]$ $\quad \forall\, \lambda \in \Lambda$ and $\forall\, \mu \in \Lambda^\circ_{\zeta_e}$.

Indeed, let P^λ be the Gram matrix of $\phi^\lambda_{\zeta_e}$. The entries of P^λ are cyclotomic integers. If we reduce further mod ℓ, then $\operatorname{rank}(P^\lambda) = \operatorname{rank}_\ell(P^\lambda)$ if $\ell \gg 0$.

Once this is shown, the equality of decomposition numbers follows from a general factorisation result due to G.–Rouquier (1997).

James' Conjecture (1990)

If $W = W(A_{n-1}) \cong \mathfrak{S}_n$, then it's enough to assume that $e\ell > n$.

Poincaré polynomial
$$P_W = \sum_{w \in W} u^{l(w)} = \prod_{1 \leqslant i \leqslant |S|} \frac{u^{d_i} - 1}{u - 1}$$

where $d_1, \ldots, d_{|S|}$ are the "degrees" of W; we have $|W| = d_1 \cdots d_{|S|}$.

In particular, P_W is a product of cyclotomic polynomials $\Phi_d \in \mathbb{Z}[u]$.

Now note: If $\Phi_d(\xi) = 0$ for some $d \geqslant 2$, then $d = e\ell^i$ $(i \geqslant 0)$.

If $W \cong \mathfrak{S}_n$, then $\{d_i\} = \{2, \ldots, n\}$ and so "$e\ell > n$" translates to:

If Φ_d divides P_W and $\Phi_d(\xi) = 0$, then $d = e$.

General version of James' Conjecture. W any finite Weyl group.

Assume that $e\ell$ does not divide any degree of W. Then
$$\dim L_\xi^\lambda = \dim L_{\zeta_e}^\lambda \qquad \forall\, \lambda \in \Lambda.$$

Theorem. G. and Rouquier (1997)

Assume that $e\ell$ does not divide any degree of W. Then

$$|\operatorname{Irr}(\mathcal{H}_\xi)| = |\operatorname{Irr}(\mathcal{H}_{\zeta_e})|.$$

Corollary. Jacon (2004)

Under the above assumptions, we have $\Lambda^\circ_\xi = \Lambda^\circ_{\zeta_e}$.

Theorem. The set $\Lambda^\circ_{\zeta_e} \subseteq \Lambda$ is explicitly known in all cases.

A_{n-1}	Dipper and James (1986)
	$\Lambda = \{\lambda \vdash n\}$ and $\Lambda^\circ_{\zeta_e} = \{\lambda \vdash n \mid \lambda \text{ is } e\text{-regular}\}$
B_n, D_n	Jacon (2003); see below
G_2, F_4, E_6, E_7, E_8	Lux, G., Müller (1991–2007); explicit tables

Idea of the proof: Use known information about decomposition numbers. The general theory of cellular algebras implies:

Lemma.

Suppose $M \in \mathrm{Irr}(\mathcal{H}_{\zeta_e})$. Let

$$\mathcal{S}(M) := \{\lambda \in \Lambda \mid [\Delta^{\lambda}_{\zeta_e} : M] \neq 0\}.$$

Then $\mathcal{S}(M)$ has a unique maximal element, μ say, with respect to $\trianglelefteq$. We have $\mu \in \Lambda^{\circ}_{\zeta_e}$ and $M = L^{\mu}_{\zeta_e}$.

Types B_n and D_n are the most interesting. Jacon's argument uses:

- ARIKI's proof (1996) of the LASCOUX–LECLERC–THIBON conjecture;
- combinatorics of crystal bases for the affine Lie algebra $A^{(1)}_{e-1}$, FODA, LECLERC, OKADO, THIBON AND WELSH (1999).

Suppose we want to verify James' conjecture for $\mathcal{H} = \mathcal{H}_A(W)$.

This essentially amounts to the computation of

- the Gram matrix of the form ϕ^λ on Δ^λ;
- the rank of that matrix under specialisation.

Recall that Δ^λ has basis $\{e_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$ and ϕ^λ is determined by

$$C^\lambda_{\mathfrak{s}',\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{t}'} = \phi^\lambda(e_\mathfrak{s}, e_\mathfrak{t})\, C^\lambda_{\mathfrak{s}',\mathfrak{t}'} \quad + \quad \text{"lower terms"}.$$

Thus, the coefficients of the Gram matrix of ϕ^λ are structure constants for $\mathcal{H}$ and, hence, can be computed in principle. Recall:

$$C^\lambda_{\mathfrak{s},\mathfrak{t}} = \text{ certain linear combination of } \mathbf{C}_w\text{'s.}$$

So, first of all, we need to be able to work out structure constants for the Kazhdan–Lusztig basis. Also possible in principle, but in E_8 ?

From now on: Joint work with Jürgen Müller.

Let $K = \mathbb{Q}(u^{1/2})$ and $\mathcal{H}_K = K \otimes_A \mathcal{H}$, a split semisimple algebra.

By the theory of cellular algebras:

$$\mathrm{Irr}(\mathcal{H}_K) = \{\Delta_K^\lambda \mid \lambda \in \Lambda\}.$$

GYOJA (1984): Every irreducible representation of $\mathcal{H}_K$ can described by a "W-graph", in the sense of LUSZTIG–KAZHDAN (1979).

NARUSE (1998) explicitly determined W-graphs for types F_4, E_6;

HOWLETT–YIN, HOWLETT (2003) did E_7, E_8.

MICHEL (2004) implemented these W-graphs in GAP.

Conclusion. Assume that W is of exceptional type.

Then, for each $\lambda \in \Lambda$, we have a collection of matrices $\rho^\lambda(T_s)$ $(s \in S)$ which afford Δ_K^λ; the coefficients of $\rho^\lambda(T_s)$ are in A.

Another try to find the Gram matrix of ϕ^λ.

Consider matrix representation (coming from W-graph):

$$\rho^\lambda \colon \mathcal{H} \to M_{d_\lambda}(A), \qquad \text{where} \qquad d_\lambda = |M(\lambda)|.$$

Let P^λ be the Gram matrix of ϕ^λ. Then the $\mathcal{H}$-invariance of ϕ^λ is equivalent to:

$$(*) \qquad P^\lambda \cdot \rho^\lambda(T_s) = \rho^\lambda(T_s)^{\text{tr}} \cdot P^\lambda \quad \forall\, s \in S.$$

Since ρ^λ defines an irreducible representation of $\mathcal{H}_K$, condition $(*)$ uniquely determines P^λ up to scalar multiples (Schur's Lemma).

$(*)$ is a system of $|S|d_\lambda^2$ linear equations for the d_λ^2 coefficients of P^λ.

This seems feasable for W of type G_2, F_4, E_6 (where $d_\lambda \leqslant 90$) but, again, it is totally unrealistic in type E_8 (where $\exists\, d_\lambda = 7168$).

Now note: Given $\rho^\lambda\colon \mathcal{H} \to M_{d_\lambda}(A)$, the map

$$\hat{\rho}^\lambda\colon \mathcal{H} \to M_{d_\lambda}(A), \qquad T_w \mapsto \rho^\lambda(T_{w^{-1}})^{\text{tr}}$$

also is a representation. Thus, $(*)$ means that P^λ is the matrix of an $\mathcal{H}$-module homomorphism between ρ^λ and $\hat{\rho}^\lambda$.

Theorem. Benson and Curtis (1972)

Every ρ^λ is of "parabolic type", that is, $\exists\, J \subseteq S$ such that

$$\dim_K\Big(\bigcap_{s\in J} \ker\big(\rho^\lambda(T_s) - u\,\mathrm{Id}_{d_\lambda}\big)\Big) = 1.$$

Let $v_1 \neq 0$ be a vector in the above intersection.

Let $\hat{v}_1 \neq 0$ be in the analogous intersection with respect to $\hat{\rho}^\lambda$.

By the above theorem, $v_1, \hat{v}_1$ are unique up to scalar.

So condition $(*)$ implies that $\hat{v}_1 = \alpha\, P^\lambda v_1$ for some scalar $\alpha \neq 0$.

Since ρ^λ irreducible (over K), we have $K^{d_\lambda} = \langle \rho^\lambda(T_w)v_1 \mid w \in W\rangle_K$.

Applying repeatedly T_s ($s \in S$), find $w_2, \ldots, w_{d_\lambda} \in W$ such that

$$\left.\begin{array}{l} v_1 \\ v_2 = \rho^\lambda(T_{w_2})v_1 \\ \quad\vdots \\ v_{d_\lambda} = \rho^\lambda(T_{w_{d_\lambda}})v_1 \end{array}\right\} \quad \text{form a basis of } K^{d_\lambda}.$$

Then

$$\left.\begin{array}{l} \hat{v}_1 \\ \hat{v}_2 = \hat{\rho}^\lambda(T_{w_2})\hat{v}_1 \\ \quad\vdots \\ \hat{v}_{d_\lambda} = \hat{\rho}^\lambda(T_{w_{d_\lambda}})\hat{v}_1 \end{array}\right\} \quad \text{also form a basis of } K^{d_\lambda},$$

and we have $\hat{v}_i = \alpha P^\lambda v_i$ for all i. Thus, can compute P^λ.

This is a version of the "**standard base algorithm**" in PARKER's **MeatAxe** programs (developped in the 1980's with the aim of constructing matrix representations of sporadic simple groups).

Actually, we cannot do this over $K = \mathbb{Q}(u^{1/2})$. Instead, specialize $u^{1/2}$ to various natural numbers, then reduce modulo various primes. Using Chinese remainder + interpolation, recover P^λ.

Computations took a wee while: more than 1 year ! But note: once a matrix P^λ is determined, it is to check if $(*)$ holds.

Theorem. G. AND MÜLLER (2007/08). W of type G_2, F_4, E_6, E_7, E_8

The general version of James' conjecture is true for W.

The dimensions $\dim L^\mu_{\zeta_e}$ and the decomposition numbers $[\Delta^\lambda_{\zeta_e} : L^\mu_{\zeta_e}]$ are explicitly known in the form of tables.