Two applications of Littelmann's path model

Susumu Ariki

Research Institute for Mathematical Sciences
Kyoto University

Lie Theory, MSRI

March 14, 2008

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Let $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix,

$$(A, \Pi = \{\alpha_i\}_{i\in I}, \Pi^\vee = \{h_i\}_{i\in I}, P, P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}))$$

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A set B is called a $\mathfrak{g}$-crystal if it is equipped with maps

$$\mathrm{wt} : B \to P, \ \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}, \ \epsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}$$

such that ...

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(5) If $\varphi_i(b) = -\infty$ then $\tilde{e}_i b = 0$ and $\tilde{f}_i b = 0$.

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A Lakshmibai-Seshadri path is a piecewise linear path $p(t) : [0, 1] \to \mathfrak{h}^*$ which change directions at $t = a_1, \dots, a_{s-1}$ and the new direction vector at $t = a_{i-1}$ is $\nu_i \in W\Lambda$, for $1 \leq i \leq s$, such that it satisfies two conditions in the next slide.

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We do not explain the definition of a_j-chain and the order $\nu_j > \nu_{j+1}$, but the condition to be a Lakshmibai-Seshadri path is as follows.

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In the rest, $\mathfrak{g} = \mathfrak{g}(A^{(1)}_{e-1})$ and consider the fundamental weights $\Lambda = \Lambda_m$, for $m \in \mathbb{Z}/e\mathbb{Z}$. Then $B(\Lambda_m)$ is realized on the set of e-restricted partitions, and it admits W-action.

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(i) $\epsilon_i(\psi(b)) = h\epsilon_i(b)$ and $\varphi_i(\psi(b)) = h\varphi_i(b)$,

(ii) $wt(\psi(b)) = hwt(b)$,

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Lemma (Kashiwara)

Let Λ be dominant integral. Then there exists a unique crystal morphism $S_h : B(\Lambda) \to B(h\Lambda)$ of amplitude h, for all $h \in \mathbb{Z}_{\geq 1}$.

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Let $\lambda \in B(\Lambda_m)$. Using $B(h\Lambda_m) \subseteq B(\Lambda_m)^{\otimes h}$, we can write

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$$S_h(\lambda)^{1/h} = \lambda^{(1)^{\otimes 1/h}} \otimes \cdots \otimes \lambda^{(h)^{\otimes 1/h}},$$

and replace $(\mu^{\otimes 1/h})^{\otimes k}$ with $\mu^{\otimes k/h}$, for any μ that appears in $\lambda^{(1)}, \dots, \lambda^{(h)}$.

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$$S_h(\lambda)^{1/h} = {\nu_1}^{\otimes a_1} \otimes {\nu_2}^{\otimes (a_2 - a_1)} \otimes \cdots \otimes {\nu_s}^{\otimes (1 - a_{s-1})},$$

where $a_0 = 0 < a_1 < \cdots < a_s = 1$ are rational numbers and $\nu_1, \dots, \nu_s$ are pairwise distinct e-restricted partitions.

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If h is sufficiently divisible then

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as above, and define π_λ to be the path given by $(wt(\nu_1), \ldots, wt(\nu_s); a_0, \ldots, a_s)$.

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as above, and define π_λ to be the path given by $(wt(\nu_1), \ldots, wt(\nu_s); a_0, \ldots, a_s)$. Then $wt(\nu_j) = w_j\Lambda_m$, for $1 \le j \le s$, and the following hold.

The path model using cores (6)

Let $\mathbb{B}(\Lambda_m)$ be the crystal of Lakshmibai-Seshadri paths.

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(1) *π_λ is a Lakshmibai-Seshadri path.*

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A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is an *e*-core if we cannot remove a hook of length *e* from λ.

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Definition

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is an e-core if we cannot remove a hook of length e from λ.

The set of e-cores is the W-orbit through the empty partition. Recall that the map $\nu \mapsto \mathrm{wt}(\nu)$, for e-cores ν, is injective. Thus the theorem gives a realization of $B(\Lambda_m)$ in terms of sequences of e-cores. Namely,

The path model using cores (7)

Path model using e-cores

$$B(\Lambda_m) \simeq \{\nu_1{}^{\otimes a_1} \otimes \nu_2{}^{\otimes(a_2-a_1)} \otimes \cdots \otimes \nu_s{}^{\otimes(1-a_{s-1})}\},$$

where $\nu_1{}^{\otimes a_1} \otimes \nu_2{}^{\otimes(a_2-a_1)} \otimes \cdots \otimes \nu_s{}^{\otimes(1-a_{s-1})} = S_h(\lambda)^{1/h}$, for e-restricted λ and sufficiently divisible h.

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Example

Let $m = 0$ and $e = 3$. Then

$$S_h((3,1^3))^{1/h} = (4,2,1^2)^{\otimes 1/3} \otimes (3,1^2)^{\otimes 2/3}.$$

In the next slide, we explain how to compute this.

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In the next slide, we explain how to compute this. Recall that if λ is an e-core, then $s_i\lambda$ is obtained by either adding all the addable i-nodes or removing all the removable i-nodes.

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$${\nu_1}^{\otimes a_1 h} \otimes {\nu_2}^{\otimes(a_2-a_1)h} \otimes \cdots \otimes {\nu_s}^{\otimes(1-a_{s-1})h} = \mu_1 \otimes \cdots \otimes \mu_h.$$

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Then we may write

$$\tilde{f}_i^h (\mu_1 \otimes \cdots \otimes \mu_h) = \tilde{f}_i^{c_1} \mu_1 \otimes \tilde{f}_i^{c_2} \mu_2 \otimes \cdots \otimes \tilde{f}_i^{c_h} \mu_h,$$

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for some non-negative integers c_i such that $\sum_{i=1}^h c_i = h$. Then, for some multiple h' of h, we have

$$S_{h'}(\tilde{f}_i\lambda)^{1/h'} = \Big((s_i{\mu_1}^{\otimes(c_1/\varphi_i(\mu_1))h'/h} \otimes {\mu_1}^{\otimes(1-c_1/\varphi_i(\mu_1))h'/h}) \otimes \cdots$$
$$\cdots\cdots \otimes (s_i{\mu_h}^{\otimes(c_h/\varphi_i(\mu_h))h'/h} \otimes {\mu_h}^{\otimes(1-c_h/\varphi_i(\mu_h))h'/h})\Big)^{1/h'}.$$

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First, take $h = 1$. Then $1/\varphi_0((3, 1^2)) = 1/3$ and

$$S_h((3, 1^3))^{1/h} = (4, 2, 1^2)^{1/3} \otimes (3, 1^2)^{2/3}$$

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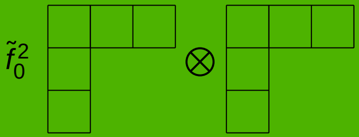

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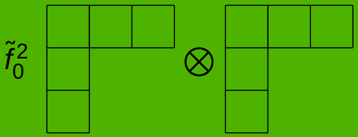

that is, $c_1 = 2$, $c_2 = 0$. Thus,

$$c_1/\varphi_0((3,1^2)) = 2/3 \text{ and } c_2/\varphi_0((3,1^2)) = 0$$

and $S_h((3,1^3))$ is equal to

$$((4,2,1^2)^{2/3}\otimes(3,1^2)^{1/3})\otimes((4,2,1^2)^0\otimes(3,1^2)^1).$$

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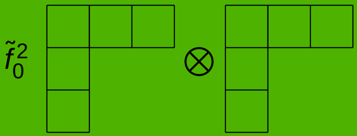

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$$((4,2,1^2)^{2/3} \otimes (3,1^2)^{1/3}) \otimes ((4,2,1^2)^0 \otimes (3,1^2)^1).$$

Thus, $S_h((3,1^3))^{1/h} = (4,2,1^2)^{1/3} \otimes (3,1^2)^{2/3}$ as before.

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Define $m(\lambda)$ by $(D^\lambda)^\tau \simeq D^{m(\lambda)}$.

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Ford and Kleshchev proved the Mullineux conjecture. Kleshchev proved the rule for a different description of the Mullineux map, and they proved that the two coincide.

The Mullineux map (3)

Lascoux, Leclerc and Thibon observed that the description by Kleshchev may be stated as follows.

Theorem

Let λ be an e-restricted partition, and view it as an element of $B(\Lambda_0)$. We write

$$\lambda = \tilde{f}_{i_n} \cdots \tilde{f}_{i_1} \emptyset,$$

for $i_1, \ldots, i_n \in \mathbb{Z}/e\mathbb{Z}$. Then

$$m(\lambda) = \tilde{f}_{-i_n} \cdots \tilde{f}_{-i_1} \emptyset.$$

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Thus, the Mullineux map is a crystal theoretic notion. The crystal graph of $B(\Lambda_0)$ in terms of e-restricted partitions is often called *Kleshchev's good lattice*.

New description of the Mullineux map (1)

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Theorem

Let λ be an e-restricted partition, and

$$(\nu_1, \ldots, \nu_s; a_0, \ldots, a_s)$$

its description in the path model using cores. Then, the description of $m(\lambda)$ in the path model is given by

$$(\nu'_1, \ldots, \nu'_s; a_0, \ldots, a_s),$$

where ν'_i is the transpose of ν_i, for $1 \leq i \leq s$.

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The block algebra labelled by ν is semisimple if and only if $|\nu| = n$. In this case, it has the unique simple module D^ν and $S^\nu = D^\nu$ holds. Hence, we have.

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Our slogan is:

The Mullineux map is ALWAYS given by transpose of partitions, if we work in the path model using cores.

The Hecke algebra of type B

Now we turn to the Hecke algebra of type B. $\mathcal{H}_n(Q, q)$ has one more generator T_0 than $\mathcal{H}_n(q) = \langle T_1, \ldots, T_{n-1} \rangle$ and the relations to be added are

$$(T_0 - Q)(T_0 + 1) = 0, \quad (T_0T_1)^2 = (T_1T_0)^2,$$

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Dipper, James and Murphy showed that the algebra is cellular. They constructed Specht modules S^λ indexed by bipartitions $\lambda=(\lambda^{(1)},\lambda^{(2)})$ of size n. Thus, S^λ has a invariant symmetric bilinear form, and we obtain D^λ by factoring out its radical.

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Dipper, James and Murphy conjectured when $D^\lambda \neq 0$.

(Q, e)-restricted bipartitions (1)

Their idea to classify simple modules resembles the highest weight theory in Lie theory. The role of the Cartan algebra is played by the commutative subalgebra $\mathcal{T}$ generated by Jucys-Murphy elements $L_1, \ldots, L_n$ defined by

$$L_1 = T_0, \quad L_{i+1} = q^{-1} T_i L_i T_i \ \text{(for } 1 \leq i < n.\text{)}$$

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Recall that a *standard bitableau of shape* λ is a bijective map $\mathbf{t}$ from the set of nodes of the bipartition λ to $[1,n]$.

Each standard bitableau $\mathbf{t}$ defines a linear character $\varphi_{\mathbf{t}}$ of $\mathcal{T}$: for $1 \le k \le n$, if $\mathbf{t}^{-1}(k)$ is the $(a,b)^{th}$ node of $\lambda^{(c)}$ then

$$\varphi_{\mathbf{t}}(L_k) = \begin{cases} Qq^{b-a} & (c=1) \\ -q^{b-a} & (c=2). \end{cases}$$

(Q, e)-restricted bipartitions (2)

In the highest weight theory, the highest weight does not appear in proper submodules. Hence, we want to consider linear characters φ of $\mathcal{T}$ that appears in S^λ but does not appear in proper submodules of S^λ.

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The Dipper-James-Murphy conjecture

D^λ is nonzero if and only if λ is (Q, e)-restricted.

The DJM conjecture is true

Jacon and I proved the following, which is the second application of the Littelmann's path model.

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Let $\mathcal{I}^{\trianglerighteq\lambda}$ be cell ideals. The form on the Specht module

$$S^\lambda \subseteq \mathcal{I}^{\trianglerighteq\lambda}/\mathcal{I}^{\triangleright\lambda}$$

is defined by product of the cellular basis (Murphy basis) elements as follows.

$$m_{\mathbf{us}}m_{\mathbf{tv}} = \langle m_{\mathbf{s}}, m_{\mathbf{t}}\rangle m_{\mathbf{uv}}.$$

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Thus, $D^\lambda = 0$ if and only if all the products of Murphy basis elements of shape λ belong to $\mathcal{I}^{\triangleright\lambda}$.

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Let $-Q = q^m$. We say that $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is *Kleshchev bipartition* if $\lambda^{(2)} \otimes \lambda^{(1)}$ belongs to the connected component of $B(\Lambda_0) \otimes B(\Lambda_m)$ that contains the empty bipartition. Then

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Theorem

$D^\lambda \neq 0$ if and only if λ is a Kleshchev bipartition.

Hence, the conjecture is equivalent to the statement that λ is (Q, e)-restricted if and only if λ is Kleshchev.

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Hence, in order to apply his theorem to our problem, we need to find the e-cores which correspond to the two direction vectors.

The path model revisited (2)

Two theorems

(1) The e-core which corresponds to the initial direction vector of the path corresponding to λ is given by Kreiman, Lakshmibai, Magyar and Weyman.

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Thank you for your attention.