3/17/08 4PM George Lusztig - "Unipotent Classes & Special Weyl Group Rep$^{\underline{n}}$'s"

$\boxed{\text{Slide 1}}$

G simple alg grp $/\mathbb{C}$.

Def$^{\underline{n}}$ $u \in G$ is unipotent if $\rho(u)$ has all eig val's 1 for any homom $\rho : G \to GL_n(\mathbb{C})$ of alg. grps.

G_u = Set of unipotent elem's in G

$G_{u}/\sim$ = " " " conj. classes in G.

The classification of unip conj. classes was obtained by Dynkin / Kostant (1950's)

Type A : Weirstrass type B, C, D : Williamson 1930's

In type A_{n-1} $G_u/\sim \;\longleftrightarrow\; \mathrm{Irr}(W)$ W = Weyl grp

Both sets are indexed by partitions of n.

$\boxed{\text{Slide 2}}$

A more direct relationship in Green (1955)

u is unipotent in PGL_n, $\mathcal{B}_u = \{\text{flags } u\text{-invariant}\}$.

$\#\{\text{Set of irred. components of } \mathcal{B}_u \text{ of top dim}\}$
$= \dim(\text{an irred rep}^{\underline{n}} \text{ of } S_n)$

Springer (1976) defined for any G an imbedding

$*$: $G_u/\sim \;\hookrightarrow\; \mathrm{Irr}(W)$

Let u unip in G $\mathcal{B}_u \subset \mathcal{B}$ full flag manifold

$H^*\mathcal{B} \xrightarrow{\Phi} H^a(\mathcal{B}_u)$ with coeff's in $\mathbb{Q}$

Take $a = 2\dim \mathcal{B}_u$. Now, W acts naturally on $H^*(\mathcal{B}) = H^a(G/\text{max torus})$ & image Φ is an irred W-module E_u. Now, $u \mapsto E_u$ is the Springer map.

<u>Questions</u>: 1) Describe image of $*$ combinatorially.

2) Describe the fcn $u \longrightarrow \underset{=b(u)}{\dim \mathcal{B}_u}$ combinatorially in terms of $*$

3) Describe the fcn $u \to \#\begin{pmatrix}\text{conn. comp. of} \\ \text{centralizer of } u \text{ in } G\end{pmatrix} =: Z(u)$ "

4) Describe the fcn $u \longrightarrow \#\begin{pmatrix}\text{conn. comp. of} \\ \text{centralizer of } \tilde{u} \text{ in } \tilde{G}\end{pmatrix} =: \tilde{Z}(u)$ "

Here $\tilde{G} \to G$ is the simply connected covering of G.

$\boxed{\text{Slide 3}}$ Examples in rank 2

Type A_2

unip	$b(u)$	$Z(u)$	$\tilde{Z}(u)$	E_u
J_3	0	1	3	1
$J_2 J_1$	1	1	1	ref. ← reflection rep^{n}
$J_1 J_1 J_1$	3	1	1	sgn

Type B_2

J_4	0	1	2	1
$J_2 J_2$	1	2	2	ref
$J_2 J_1 J_1$	2	1	2	Idem. ref ← $a_E = 1$ $b_E = 2$ ⇒ this is not special (See next pg for def^{n})
$J_1 J_1 J_1 J_1$	4	1	1	Sgn

Let $\tilde{I} = I \cup \{0\}$. For $i \in \tilde{I}$ Let $s_i : X \to X$
$s_i(x) = x - \langle \check{\alpha}_i, x \rangle \alpha_i$. W sbgrp of $GL(X)$ gener by
s_i $(i \in I)$ of s_i $(i \in \tilde{I})$ s_i $(i \in \tilde{I})$ satisfy
rel's of affine Weyl grp dual to R.
$\tilde{A} = \{ J \subsetneq \tilde{I} \}$. For $J \in \tilde{A}$ Let W_J = subgr of W,
gen by $\{s_i, i \in J\}$ a finite Weyl Grp.

<u>Truncated induction</u>: Let $J \in \tilde{A}$ $E_1 \in \mathrm{Irr}(W_J)^+$
Then $\exists !$ $E \in \mathrm{Irr}\, W$ s.t. $(E : \mathrm{Ind}_{W_J}^{W} E_1) > 0$

$b_E = b_{E_1}$.
Moreover, $(E : \mathrm{Ind}_{W_J}^{W} E_1) = 1$, $E \in \mathrm{Irr}\, W^+$
Set $E = j_{W_J}^{W} E_1$

$\boxed{\text{Slide 6}}$ <u>Semispecial Rep$^{\underline{n}}$ of W</u> $E \in \mathrm{Irr}(W)$ is semispecial
if $\exists$ $J \subsetneq \tilde{I}$ & $E_1 \in \mathcal{S}_{W_J}$ s.t. $E = j_{W_J}^{W} E_1$.

<u>Let</u> $\bar{\mathcal{S}}_W$ = set <u>of</u> semi special repn's of W.

Note $\mathcal{S}_W \subset \bar{\mathcal{S}}_W$

eg. type A_n: $\mathcal{S}_W = \bar{\mathcal{S}}_W = \mathrm{Irr}\, W$
type B_2: $\mathrm{Irr}\, W = \{1, \mathrm{refl}, \varepsilon, \varepsilon', \mathrm{sgn}\}$
$\mathcal{S}_W = \{1, \mathrm{ref}, \qquad \mathrm{sgn}\}$
$\bar{\mathcal{S}}_W = \{1, \mathrm{ref}, \varepsilon, \mathrm{sgn}\}$

ε is obtained as $j_{A_1 \times A_1}^{W}(\mathrm{sgn})$

Thm 1 : The image of the Springer imbedding $G_u/_\sim \hookrightarrow Irr(W)$ is exactly $\overline{\mathcal{S}}_W$.

$\boxed{\text{Slide 7}}$ Another example

type G_2 $Irr\, W = \{1, ref, ref', \varepsilon, \varepsilon', sgn\}$

$\mathcal{S}_W = \{1, ref, \quad sgn\}$

$\overline{\mathcal{S}}_W = \{1, ref\ ref', \varepsilon \quad sgn\}$

ref' obtained as $j^W_{A_1 \times A_1}(sgn)$

ε obtained as $j^W_{A_2}(sgn)$

$\boxed{\text{Slide 8}}$ The fcn $u \to b(u) = \dim \mathcal{B}(u)$

Claim $b(u) = b_E$ where $E = E_u \in \overline{\mathcal{S}}_W$ is the irrep of W corresp to u.

The fcn $u \to z(u)$

Claim $z(u) = \max\{f_{E_1}\}$ $(J, E_1),\ J \subsetneq \tilde{I}$, $E_1 \in \mathcal{S}_{W_J}$, $E = j^W_{W_J}(E_1)$

Def $\Omega = \{w \in W;\ w(\alpha_i) = \alpha_{\underline{w}(i)}\ \forall i \in \tilde{I}\}$ for some nec. unique permutation $\underline{w}: \tilde{I} \to \tilde{I}$

: commutative s.g. of W

Remark Ω is isomorphic to the center of $\tilde{G}$

$\boxed{\text{Slide 9}}$ The fnc $u \to \frac{\tilde{z}(u)}{z(u)}$

Let $E \in \overline{\mathcal{S}}_W$. Let $Z_E = \{(J, E_1);\ J \subsetneq \tilde{I}\ \ j^W_{W_J} E_1 = E,\ E_1 \in \mathcal{S}_{W_J}\}$

Note $Z_E \neq \emptyset$ $\quad \underline{a}_E = \max_{(J, E_1) \in Z_E} \mathcal{S}_{E_1}$

Let $Z_E^* = \{(J, E_1) \in Z_E \mid f_{E_1} = a_E\}$ Note: $Z_E^* \neq \emptyset$.

Now, Ω acts on Z_E $w: (J, E_1) \to (\underline{w}(J), {}^w E_1)$

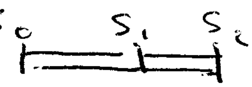
This restricts to an action of Ω on Z_E^*.

Let Ω_{JE_1} = stabilizer of $(J, E_1) \in Z_E^*$ in Ω.

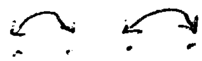
Claim: $\dfrac{\tilde{Z}(u)}{Z(u)} = \max\limits_{(J,E)\in Z_E^*} |\Omega_{JE_1}|$ where $E = E_u$

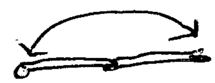
$\boxed{\text{Slide 10}}$ (Sp_4)

$\tilde{I} = \{0, 1, 2\}$. $(s_0 s_1)^4 = (s_1 s_2)^4 = (s_0 s_2)^2 = 1$



$W = \left\{ \begin{array}{l} \rho: \{1, 2, 2', 1'\} \xrightarrow{\sim} \{1, 2, 2', 1'\} \\ \rho \text{ commutes with } 1 \leftrightarrow 1',\ 2 \leftrightarrow 2' \end{array} \right\}$

$s_0 =$  $s_1 =$ 

$s_2 =$  Ω acts as 

unip of type J_4 $E_u = 1$ $j_\emptyset^W(1) = 1$ $(\emptyset, 1)$ is Ω-stable

$\tilde{Z}(u)/Z(u) = 2$

" " " $J_2 + J_2$ $E_{u} = \text{ref}$ $j_{\{1,2\}}^W \text{ref} = \text{ref}$ $\{1,2\}$ is not Ω-stable

$j_{\{1\}}^W \text{sgn} = \text{ref}$

$\tilde{Z}(u)/Z(u) = 1$ by $\rho_{\text{sgn}} \neq Z(u)$
 $\rho_{\text{sgn}} = 1$, $Z(u) = 2$

" " " $J_2 + J_1 + J_1$ $E_u = 1\text{-dim} = \varepsilon$

$j_{\{0,2\}}^W \text{sgn} = \varepsilon$ $\{0,2\}$ is Ω-stable

$\tilde{Z}(u)/Z(u) = 2$

" " " $J_1 J_1 J_1 J_1$ $E_u = \text{sgn}$ $j_{\{1,2\}}^W \text{sgn} = \text{sgn}$ $\{1,2\}$ not Ω-stable

$\tilde{Z}(u)/Z(u) = 1$

$\boxed{\text{Slide 11}}$ $u \in G$ special unipotent $\longleftrightarrow$ corresp. irred rep in $\mathcal{S}_W$ is in $\mathcal{S}_W$ (special)

Let G^* be simply conn. grp K of type Langlands dual to that of G.

Def$^{\underline{n}}$ $g \in G^*$ is special if $g = g_s g_u$ (Jordan decomp) & g_u is special unipotent in $Z_{G^*}(g_s)$.

Let $G^*_{spec} = \{ g \in G^* \mid g \text{ special} \}$

RK| We have the canonical maps

$$\begin{tikzcd}
G^*_{spec}/\text{conj} \arrow[r] \arrow[d, "\Phi"'] & \{(J, E_1) \mid J \subsetneq I, \; E_1 \in \mathcal{S}_{W_J}\} \arrow[d, "\text{trunc. ind.}"] \\
G_{unip}/\text{conj} \arrow[r, "\sim"] & \overline{\mathcal{S}}_W
\end{tikzcd}$$

Φ is surj.

End of Lecture