3/18/08 11AM – Jan Schroeer – "Dual semicanonical bases & cluster algebras".

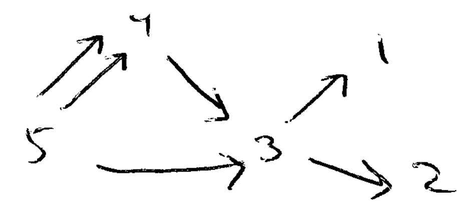
- This is his 1^{st} visit to the U.S.
- Joint work with C. Geiß & B. Leclerc

Semicanonical bases introduced by Lusztig 1992, 2000

Cluster algebras " " Fomin / Zelevinsky 2000

$\boxed{\S 1}$ Quivers $Q = (\underset{\{1,\dots,n\}}{I}, \underset{\text{arrows}}{Q_1})$ fin. quiver, no oriented cycles

[if $i \to j$, then $i > j$] e.g.



$0 \le a < b$ $a, b \in \mathbb{N}$. ($b \le$ some bound if Q, dynkin)

$f_{i,\underbrace{[a+1,b]}_{l-1}} \cdot f_{i,\underbrace{[a,b-1]}_{l-1}}$ "lengths below each bracket [,]"

$$= f_{i,\underbrace{[a,b]}_{l}} \cdot f_{i,\underbrace{[a+1,b-1]}_{l-2}} - \prod_{\substack{j \to i \\ \text{in } Q_1}} f_{j,\underbrace{[a+1,b]}_{l-1}} \prod_{\substack{i \to k \\ \text{in } Q_1}} f_{k-1,\underbrace{[a,b-1]}_{l-1}} \quad (*)$$

$f_{i,[a,b]}$ is a rational fcn in $f_{l,[c,c]}$.

$(*)$ is
- generalized determinental identity
- a multiple formula for some dual semican. basis vectors
- cluster exchange relation

Remark $f_{i,[c,d]} = 1$ if $c > d$.

$Q \quad 3 \rightrightarrows 2 \to 1$

$\boxed{\text{Example of Thm Below}}$

$$f_{3,[0,2]} = f_{3,[2,2]} f_{3,[1,1]} f_{3,[0,0]} - f_{3,[2,2]} f_{2,[0,0]}^{2}$$
$$- f_{2,[1,1]}^{2} f_{3,[0,0]} + 2 f_{2,[1,1]} f_{2,[0,0]} f_{3,[1,1]} f_{1,[0,0]}$$
$$- f_{3,[1,1]}^{3} f_{1,[0,0]}^{2}$$

Thm: $f_{i,[a,b]}$ is a polynomial in $f_{l,[c,c]}$

(Proof) • rep$^{\underline{n}}$ theory of projective algebras
• dual semican. bases
• cluster algebra #'s

eg. $Q \quad 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ $\qquad b \leq i$

$f_{i,[a,b]}$ is a minor of
$$\begin{pmatrix} 1 & f_{n,[n-1,n-1]} & \cdots & f_{1,[0,0]} \\ & 1 & \ddots & f_{2,[0,0]} \\ & & \ddots & \vdots \\ & & & f_{n,[0,0]} \\ & & & 1 \end{pmatrix}$$

Meanwhile in the Kac-Moody (KM) universe:

$Q \rightsquigarrow \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}^+$ symmetric K-M Lie alg.

$$\begin{array}{ccccc} & & \mathcal{P} \text{ (a PBW-basis)} & & \\ & & \cap & & \\ \text{(the canonical basis) } \mathcal{B} & \subset & \mathcal{U}(n^+) & \supset & \mathcal{S} \text{ (semi canon. basis)} \end{array}$$

↑ (envel. alg) is a Hopf alg.

<u>Dual Situation</u>: For simplicity, Q is not Dynkin.

For $t \geq 1$, set $W := (s_n \cdots s_1)^{t+1}$.

$\mathcal{U}(n^+)^*$ is the graded dual of $\mathcal{U}(n^+)$. It is a commutative alg.

$$\begin{array}{ccccc} \mathcal{U}(n^+)^* & \supset & \mathcal{S}^* & \supset & \mathcal{S}^* \text{ (rigid)} \\ \cup & & \cup & & \cup \\ \boxed{A_W} & \supset & \mathcal{S}^*_W & \supset & \mathcal{S}^* \text{ (cluster monomial)} \\ \cup & & & & \cup \\ \mathcal{P}^*_W & & & & \mathcal{S}^*_W \text{ (cluster variables)} \\ & & & & \rotatebox{90}{$\in$} \\ & & & & f_{i,[a,b]} \end{array}$$

This subalgebra equals $\mathbb{C}[f_{\ell,[c,c]}]$ poly. ring where $1 \leq \ell \leq n$ & $0 \leq c \leq t$

$\mathcal{P}^*_W$ ← dual PBW basis elem's "monomials in $\mathbb{C}[f_{\ell,[c,c]}]$"

$f_{i,[a,b]}$ defined in ⊛

<u>Problem</u>: Base change $\mathcal{S}^*_W \longmapsto \mathcal{P}^*_W$

This we cannot do. It is too hard.

But what we can do is pass to the cluster monomials, as in the following theorem.

<u>Thm</u> Base Change

$$\mathcal{S}^*_W \begin{pmatrix}\text{cluster}\\ \text{monomials}\end{pmatrix} \longmapsto \mathcal{P}^*_W$$

Q	W	$\mathcal{S}^*_W$ (Cluster monomials)
A_5	w_0	1-1 corresp to $\left\{\begin{array}{l}\text{real Schur roots} \\ \text{in Elliptic } E_8^{(1,1)}\end{array}\right\}$
D_4	w_0	―― " ―― $E_6^{(1,1)}$
Q general (non Dynkin)	$W = (s_n \cdots s_1)^2$	$\left\{\begin{array}{l}\text{real Schur roots} \\ \text{of } \mathfrak{g} \mapsto Q\end{array}\right\}$ (Proof) Use [BMRRT], [C,K], [HU]

A <u>Cluster</u> in A_W is a subset $\{b_1, \ldots, b_r\}$ $r = (t+1)n$ = length of W

$\{b_1, \ldots, b_r\} \subset \mathcal{S}^*_W$ (Cluster Variables).

<u>Cluster Monomials</u> are $b_1^{n_1} b_2^{n_2} \cdots b_r^{n_r}$ $(\forall\, n_i \geq 0)$

<u>Theorem</u>: ① $\{b_1, \ldots, b_r\}$ cluster $\Longrightarrow b_1^{n_1} b_2^{n_2} \cdots b_r^{n_r} \in \mathcal{S}^*_W$

② A_W is a cluster alg (in the sense of Fomin / Zelevinsky)

$\mathcal{S}^*_W := A_W \cap \mathcal{S}^*$ $\mathbb{C}$-basis of A_W

$A_W \cong \mathbb{C}[N(w)]$

<u>Initial Seed</u> $\left(\Gamma_W, \left\{ f_{i,[a,t]} \,\middle|\, \begin{array}{l} 1 \leq i \leq n \\ 0 \leq a \leq t \end{array} \right\}\right)$

<u>Defn</u> (by example) Q $3 \rightrightarrows 2 \longrightarrow 1$ $t = 2$

See graph on next page.

Initial Seed (example)

$$\begin{tikzcd}
f_{3,[2,2]} & f_{3,[1,2]} \arrow[l] \arrow[dr, shift left] \arrow[dr, shift right] & f_{3,[0,2]} \arrow[l] \\
f_{2,[2,2]} \arrow[u, shift left] \arrow[u, shift right] \arrow[dr] & f_{2,[1,2]} \arrow[l] \arrow[u, shift left] \arrow[u, shift right] \arrow[dr] \arrow[ul, leftarrow, shift left] \arrow[ul, leftarrow, shift right] & f_{2,[0,2]} \arrow[l] \arrow[u, shift left] \arrow[u, shift right] \\
f_{1,[2,2]} \arrow[u] & f_{1,[1,2]} \arrow[l] \arrow[u, "a"'] & f_{1,[0,2]} \arrow[l] \arrow[u, "a"']
\end{tikzcd}$$

$$(\Gamma, \{f_1, \dots, f_r\}) \xmapsto{\mu_K} \left(\mu_K(\Gamma), \{f_1, \dots, f_r\} \text{ but delete } f_K \text{ \& replace it by } f_K^*\right)$$

seed

$1 \le K \le r-n$

μ_K: Fomin/Zelevinsky Mutation

where $$f_K^* = \left(\prod_{\substack{k \to i \\ \text{in } \Gamma}} f_i + \prod_{\substack{j \to k \\ \text{in } \Gamma}} f_j\right) \cdot \frac{1}{f_K}$$

<u>Theorem</u>: $f_K^* \in A_W$ "$=$" $\mathbb{C}[f_{\ell,[c,c]}]$ i.e. You always get polynomials!

Iterated Mutations will give you many seeds, cluster variables. (Computer example will later show this).

$A := \mathbb{C}\overline{Q}/(c)$ projective algebra

where $Q = 1 \leftarrow 2 \leftarrow 3$, then $\overline{Q} = 1 \underset{a}{\overset{a^*}{\rightleftarrows}} 2 \underset{b}{\overset{b^*}{\rightleftarrows}} 3$.

$S_1, \dots, S_n$ are 1-dim'l simple Λ-modules.

$\underline{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ $\quad \Lambda_{\underline{d}}$ = affine varieties of Λ-modules with dim. vector $\underline{d}$

$\underline{d}=(d_1,d_2,d_3)$ $\quad \left\{ \mathbb{C}^{d_1} \underset{f_a}{\overset{f_{a^*}}{\rightleftarrows}} \mathbb{C}^{d_2} \underset{f_b}{\overset{f_{b^*}}{\rightleftarrows}} \mathbb{C}^{d_3} \;\middle|\; \begin{array}{l}\text{Satisfying}\\ \text{relation } c\\ \text{\& nilpotentcy}\end{array} \right\}$

$\mathrm{Irr}(\Lambda_{\underline{d}}) = \left\{ \begin{array}{c}\text{irred}\\ \text{comp}\end{array} \right\}$ $\quad f_a f_{a^*} = 0 = f_{a^*} f_a - f_b f_{b^*}$

$$f_{b^*} f_b = 0$$

If $X \in \Lambda_{\underline{d}}$, then

$$\delta_X : I^{\underline{d}} \longrightarrow \mathbb{C}$$

$$\underline{i}=(i_1,\dots,i_n) \longmapsto \chi(\underbrace{\mathcal{F}_{\underline{i},X}}_{\text{comp. series of } X \text{ of type } \underline{i}})$$

Euler Characteristic

$\mu^*_{\underline{d}} = \langle \delta_X \mid X \in \Lambda_{\underline{d}} \rangle_{\mathbb{C}}$ $\quad$ & $\left(\mu^* = \bigoplus_{\underline{d}\in\mathbb{N}^n} \mu^*_{\underline{d}}\right) \cong \mathcal{U}(n^+)^*$

dual Lusztig

$\mathcal{Y}^* = \left\{ \delta_X \;\middle|\; \begin{array}{l}X \text{ generic}\\ \text{in } Z\end{array} \text{ for } Z \in \mathrm{Irr}(\Lambda_{\underline{d}}) \ \underline{d}\in\mathbb{N}^n \right\}$

$\delta_X \cdot \delta_Y = \delta_{X\oplus Y}$. $\quad f_{l,[c,c]} = \delta_{\tau_Q^c(I_i)}$

where $\tau_Q^c(I_i)$ are indec. $\mathbb{C}Q$-modules with dim vectors

$\Delta_w^+ = \{\alpha \in \Delta^+ \mid w(\alpha) < 0\}$

Computer Software Link www.math.jussieu.fr/~keller/quivermutation

Bernhard Keller's site

End of Lecture