Sortable elements – Beyond Finite Type

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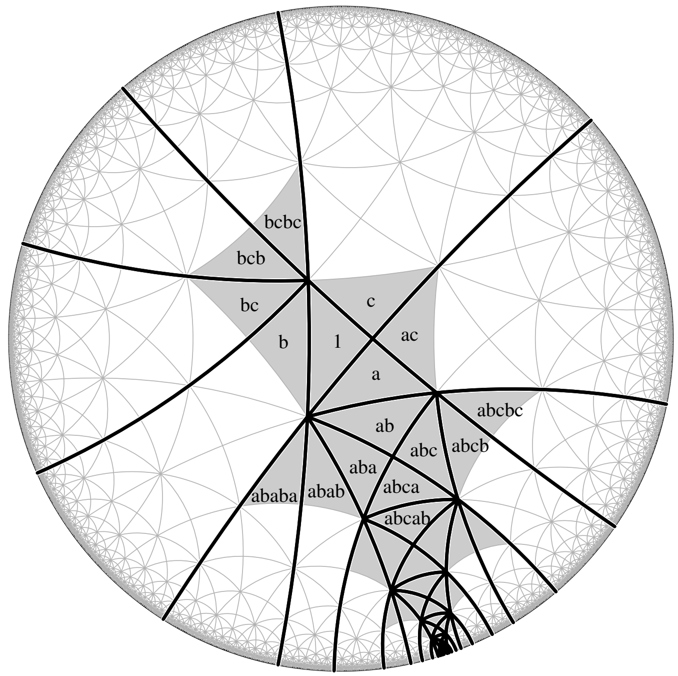


Joint work with Nathan Reading. Based on *Cambrian Fans* (arXiv:math.CO/0606201, *JEMS* to appear) and *Sortable Elements in Infinite Coxeter Groups* (coming soon!)

If W is a finite Coxeter group and c is a Coxeter element (to be defined soon), the following objects are equinumerous:

1. c-clusters
2. c-noncrossing partitions
3. W-nonnesting partitions
4. c-sortable elements

Nathan Reading introduced sortable elements to link clusters and noncrossing partitions. I will discuss sortable elements, clusters and noncrossing partitions. Nonnesting partitions are a complete mystery to us.

Sortable Elements

Let $c = s_1 \cdots s_n$. We say that $w \in W$ is c-sortable if it has a reduced word of the form

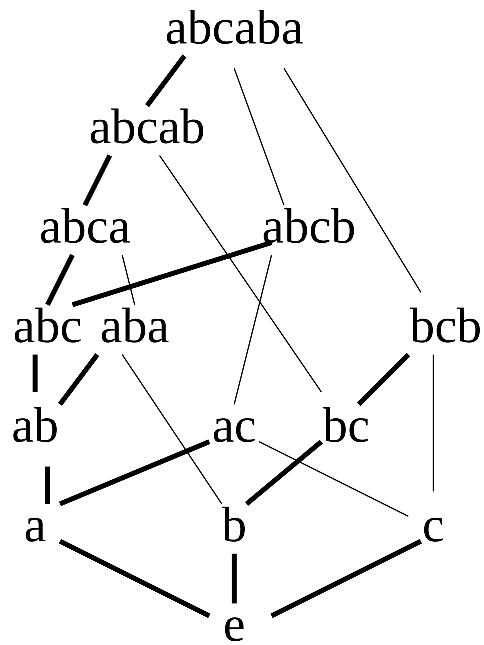
$$w = \left(s_{i^1_1} s_{i^1_2} \cdots s_{i^1_{r_1}}\right) \cdot \left(s_{i^2_1} s_{i^2_2} \cdots s_{i^2_{r_2}}\right) \cdots \left(s_{i^k_1} s_{i^1_2} \cdots s_{i^k_{r_k}}\right)$$

where

$$1 \leq i^j_1 < i^j_2 < \cdots < i^j_{r_j} \leq n$$

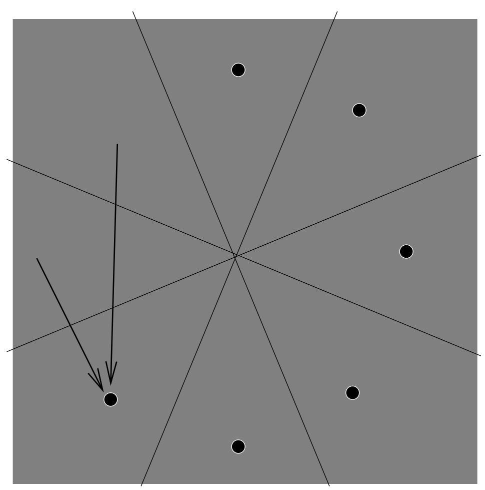
and

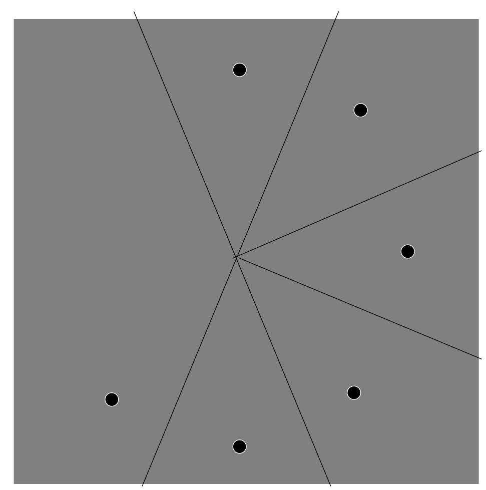
$$\cdots \supseteq \{i^j_1, \cdots, i^j_{r_j}\} \supseteq \{i^{j+1}_1, \cdots, i^{j+1}_{r_{j+1}}\} \supseteq \cdots .$$



The abc-sortable elements in A_3. The reduced words of the previous slide can be read along the bold edges.

Facts from Nathan Reading's earlier work: Let W be a finite Coxeter group. The c-sortable elements form a sublattice of W. Therefore, for each $w \in W$, there is a unique maximal element among those c-sortable elements which are less than w. We denote this element by $\pi_{\downarrow}^{c}(w)$. The map $\pi_{\downarrow}^{c}$ is a map of lattices. This quotient lattice is called the c-Cambrian lattice. We call the equivalence relation that collapses fibers of $\pi_{\downarrow}^{c}$ the c-Cambrian congruence. Thought of as a union of cones in the hyperplane arrangement, each fiber of $\pi_{\downarrow}^{c}$ is itself a cone.





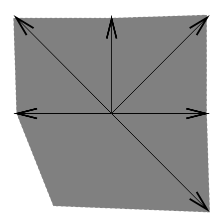
Clusters

Let W be finite, or simply-laced. The following is known in those cases, and should generalize to all W:

The set of almost positive roots is

$$\Phi_{\geq -1} = \Phi_+ \cup \{-\alpha_1, -\alpha_2, \ldots, -\alpha_n\}.$$

There is a compatibility relation on $\Phi_{\geq -1}$, defined from c. The abstract cluster complex is the simplicial complex on $\Phi_{\geq -1}$ whose faces are the compatible sets. Maximal compatible sets are called clusters; every cluster has size n. We can map the abstract cluster complex to V by sending $[\alpha]$ to α. This map is injective; we will call the image the "cluster fan".



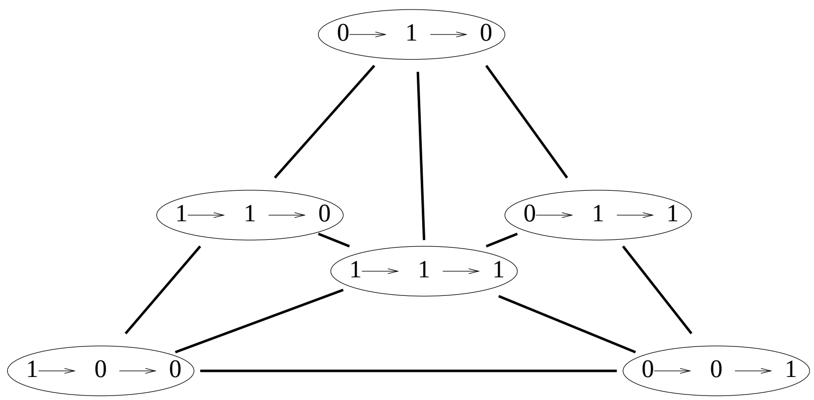
Compatibility *via* Cluster Algebras:

If W is crystallographic, then we can choose a symmetrizable, integral Cartan matrix A. Use c to anti-symmetrize A. Build a cluster algebra from the anti-symmetrized matrix; this lets us find the abstract cluster complex as a simplicial complex whose vertices are the cluster variables. Cluster variables give rise to almost positive roots by taking the denominator vector.

Note many subtleties: is the cluster variable determined by cluster variable? Is the map from the abstract cluster complex into V injective? To the best of my knowledge, these things are currently only known for W finite or simply-laced, but experts believe this should work in all crystallographic cases.

Compatibility *via* Quivers

In the simply-laced case, use c to build a quiver. (Just draw the oriented diagram.) Compatability becomes an Ext-vanishing condition. Given a dimension vector α, the cluster cone in which α lies encodes the generic decomposition of a quiver representation of dimension α.



This approach is, in principle, combinatorial (by results of Schofield, and Derksen and Weyman).

Noncrossing partitions

Recall the set of reflections T. Define $\ell_T(w)$ to be the shortest length of a T-word for w. Define the T-order by

$$u \leq_T v \text{ if and only if } \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v).$$

For W finite: W acts on the T-order by conjugation, and all of the maximal elements form a single orbit. A coxeter element c is always T-maximal, we define the c-noncrossing partitions to be the interval $[e, c]_T$. This is not just a poset, but a graded lattice.

For W finite: Associate to each noncrossing partition the subspace of V^* that it fixes. Brady and Watt showed that this is an injective map, which carries $\leq_T$ to $\supseteq$ and takes the grading to codimension. Call the image of this map the noncrossing subspaces.

The poset $[e, c]_T$ can be defined for any W, but it may not be the right definition when W is infinite.

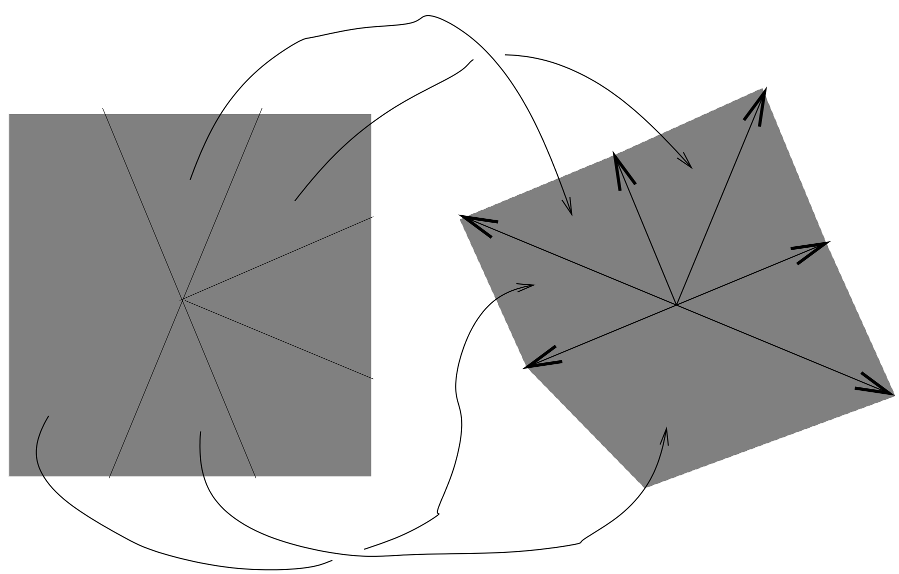
Our fundamental objects are polyhedral cones, but they live in different spaces:

• c-clusters are cones in V, defined by the span of certain roots.

• c-noncrossing subspaces are certain subspaces of V^*.

• Elements of the c-Cambrian lattice, which are in bijection with c-sortable elements, are cones in V^*. They are defined by removing walls from the hyperplane arrangement.

Theorem (Reading-S., *Cambrian Fans*)**.** *Let W be a finite Coxeter group. The cones of the c-Cambrian fan are all simplicial, and the simplical complex they form is combinatorially isomorphic to the c-cluster complex. We have a very explicit description of the walls of each of these simplicial cones.*

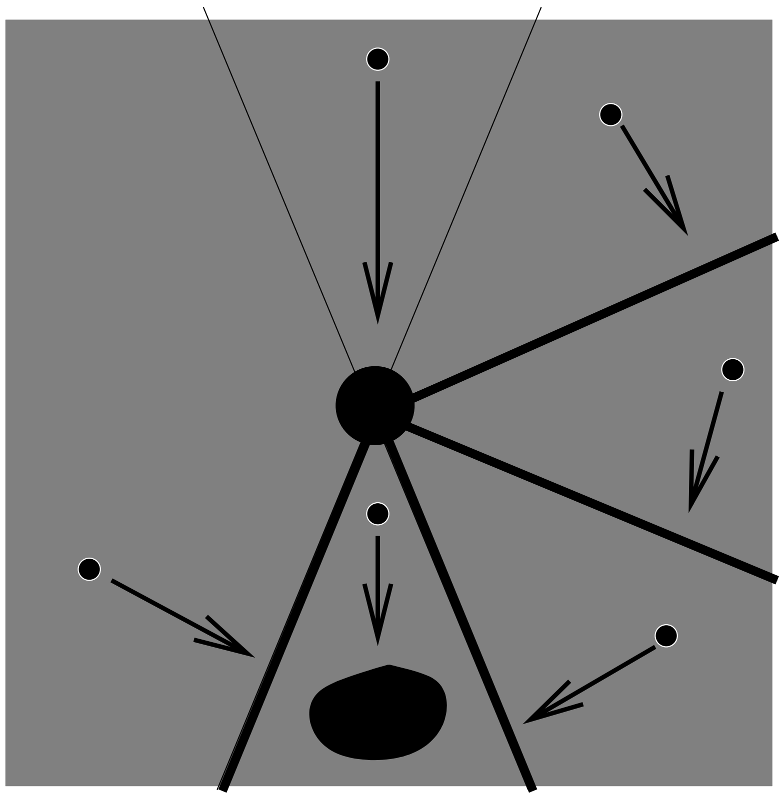


Implicit in the relation between the cluster fan and the Cambrian fan is a rule for taking a root α_t and writing down the corresponding ray ρ_t of the Cambrian fan. Explicitly, for every reflection t, there is at most one c-sortable element $j(w,t)$ such that

1. $j(w,t)$ covers precisely one element of W (i.e. $j(w,t)$ is join-irreducible) and
2. the element $j(w,t)$ covers is $tj(w,t)$.

Theorem. *The ray ρ_t is the ray of* $\mathrm{Cone}_c(j(w,t))$ *opposite the face* $H_t \cap \mathrm{Cone}_c(j(w,t))$*.*

Theorem (Reading-S. *Cambrian Fans*). *The map sending each cone in the Cambrian fan to its "bottom face" is a bijection from Cambrian congruence classes to noncrossing subspaces.*



As soon as we written the Cambrian fan paper, we began to discover sortable elements and the Cambrian fan showing up in other places.

In cluster algebras

In *Cluster Algebras IV*, Fomin and Zelevinsky introduce a vector called the g-vector, assigned to every cluster variable. Just as denominator vectors give the cluster fan, g-vectors give what I'll call the g-fan.

Theorem (Reading and S. assuming a conjecture of Fomin and Zelevinsky; Yang and Zelevinsky unconditionally (this morning's talk!)). *When W is finite, the Cambrian fan is the g-fan.*

In quiver theory

Theorem (Ingalls and Thomas, *Noncrossing partitions and representations of quivers*)**.** *Let W be a finite Coxeter group. Let K be a set of reflections. Then K is the set of inversions of a c-sortable element if and only if K corresponds to the indecomposable objects in a torsion class of the corresponding quiver.*

Theorem (follows from work of Weyman)**.** *Let β be a positive root and let $\sigma \in V^*$ be such that $\langle \sigma, \alpha_i \rangle$ is an integer for each α_i. Assume that σ is in the interior of the Tits cone (automatic if W is finite). Then the following are equivalent:*

1. *There is a wall of the c-Cambrian fan, containined in $\beta^\perp$ and containing α.*
2. *There is a semiinvariant, of degree σ, of β-dimensional representations of the c-quiver.*

What do we know outside finite type?

Theorem (Reading and S.)

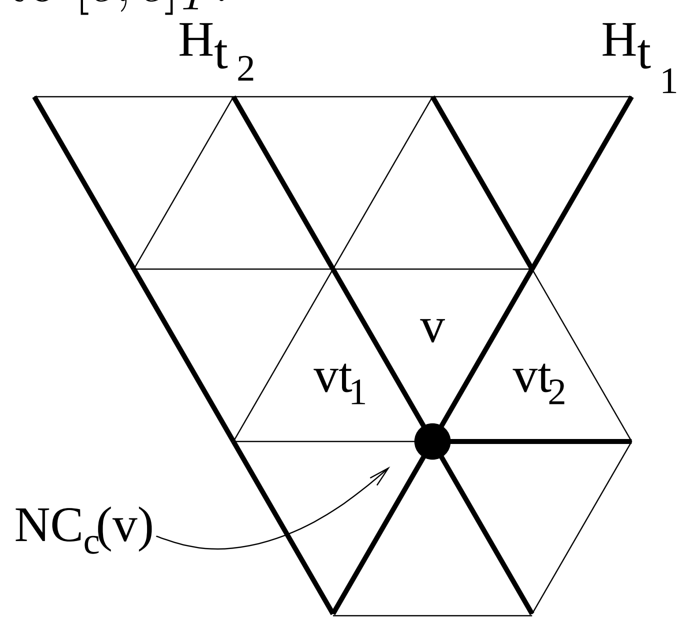
Let W be an Coxeter group. Sortable elements are defined as before. Then sortable elements form a sublattice of W in the following sense: If A is a nonempty set of sortable elements then $\bigwedge A$ is sortable; if A is a bounded above set of sortable elements then $\bigvee A$ is sortable.

Define $\pi_\downarrow$ as before. Then $\pi_\downarrow$ is a map of lattices in the following sense: If A is nonempty then $\bigwedge \pi_\downarrow(A) = \pi_\downarrow(\bigwedge A)$; if A is bounded above then $\bigvee \pi_\downarrow(A) = \pi_\downarrow(\bigvee A)$.

For each c-sortable v, there is a simplicial cone $\mathrm{Cone}_c(v)$ in V^* such that $\pi_\downarrow^{-1}(v)$ is $\mathrm{Cone}_c(v) \cap \mathrm{Tits}$. We (tenatively) define the c-Cambrian fan to be the fan $\bigcup C_c(v)$.

What about noncrossing partitions?

Theorem (Reading and S.) Let v be a c-sortable element. Let $\mathrm{NC}_c(v)$ be the subspace spanned by the bottom face of $\mathrm{Cone}_c(v)$. Then $v \mapsto \mathrm{NC}_c(v)$ is an injection from c-sdortable elements to subspaces of V^*. If $t_1 v$, $t_2 v$, $\ldots$, $t_r v$ are the elements of W covered by v, then $\mathrm{NC}_c(v) = \bigcap H_{t_i}$ and we can order the t_i so the $t_1 t_2 \ldots t_r \leq_T c$. The map $v \mapsto t_1 t_2 \cdots t_r$ is an injection from c-sortable elements to $[e, c]_T$.



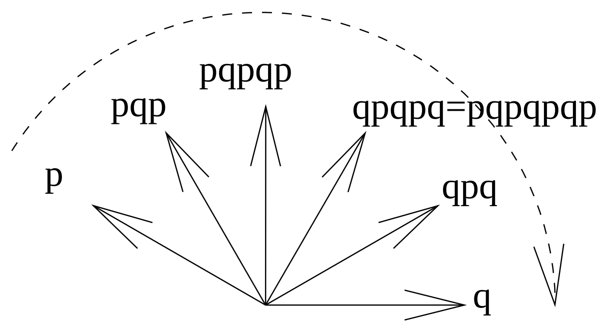
Sketch of Proof: Our main techinical tool is a "pattern-avoidance" description of sortable elements. In type $A_{n-1} = S_n$, with $c = s_1 s_2 \cdots s_{n-1}$, an element $a_1 a_2 \cdots a_n$ is c-sortable if and only if it is 312-avoiding.

In other words, for each parabolic subgroup $W' \cong A_2$ with generators $p = (ij)$ and $q = (jk)$, $i < j < k$, we are requiring that the intersection of the inversion set of w with W' is **not** $\{(ik), (jk)\} = \{qpq, q\}$.

More generally, given a Coxeter group W and Coxeter element c, for every rank two parabolic subgroup W', we give an ordering

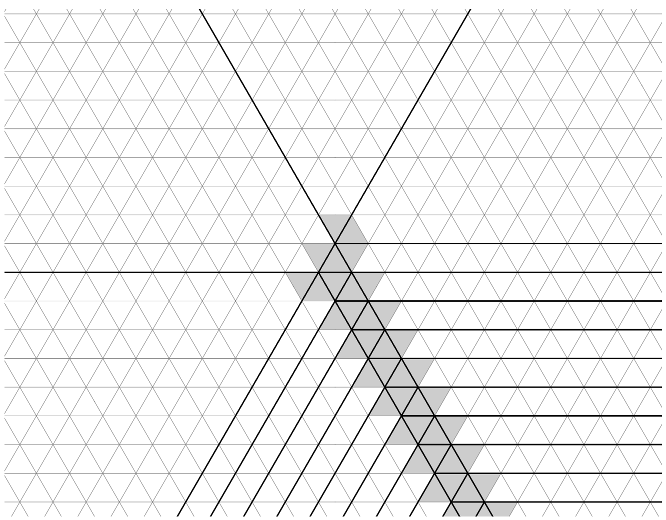
$$p,\ pqp,\ pqpqp,\ \ldots,\ qpqpq, qpq, q$$

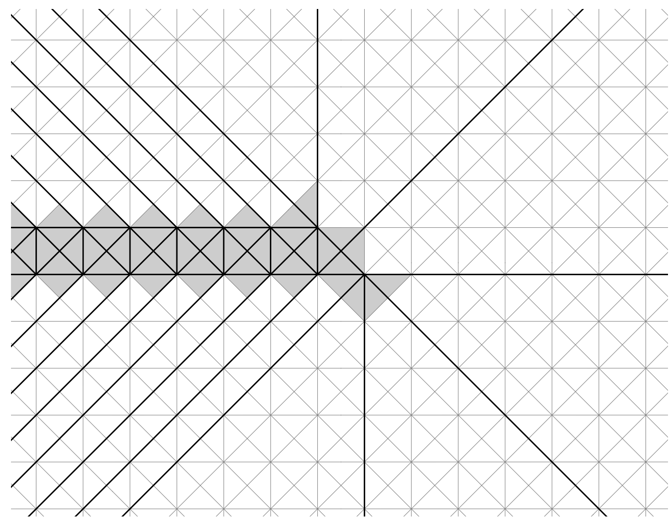
of the reflections in W'. Then c is aligned if and only if, for every W', the intersection of W' with the inversions of w is either $\{\}$, $\{q\}$ or of the form $\{p, pqp, pqpqp, \ldots\}$.



The ordering of the reflections of W' is obtained by restricting a skew-symmetric form ω_c to the two-dimensional space spanned by the roots of W'. In the simply laced case, which relates to quiver theory, ω_c is the anti-symmetrization of the Euler-form.

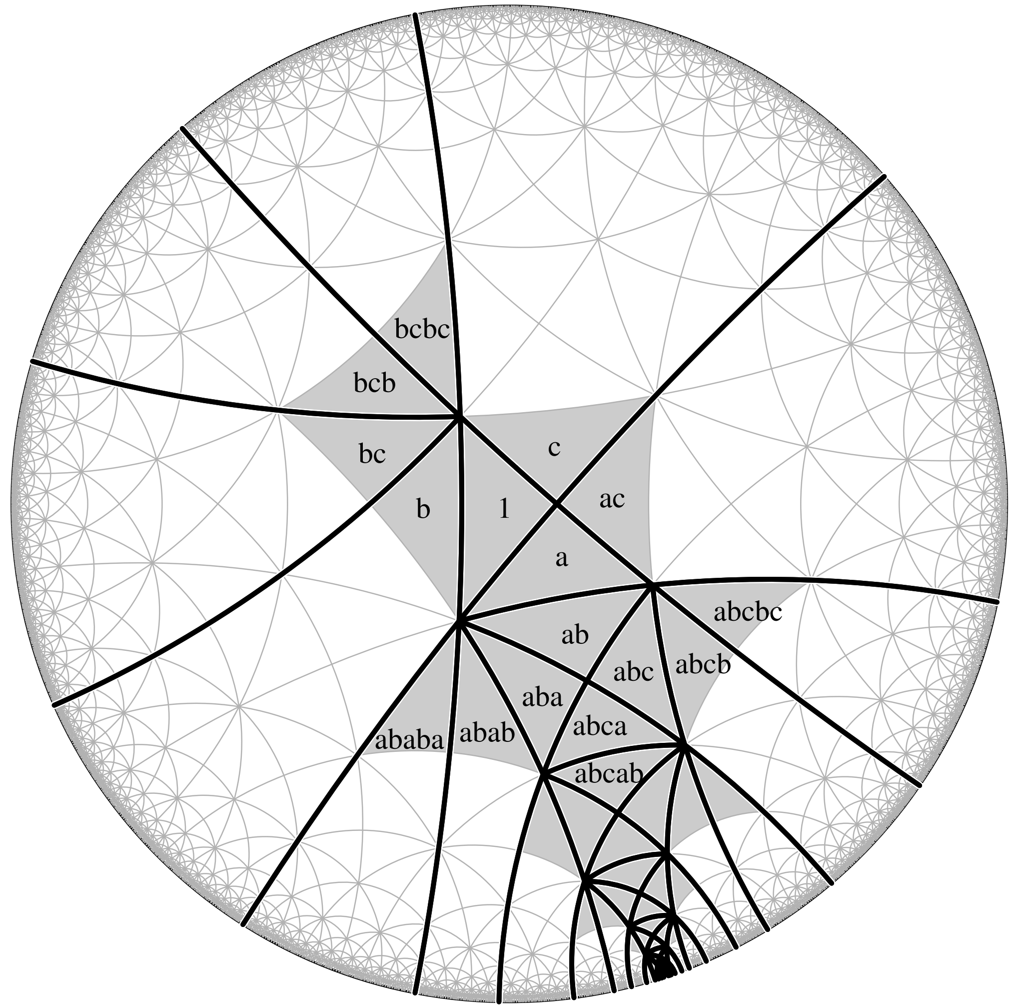
Some Pictures





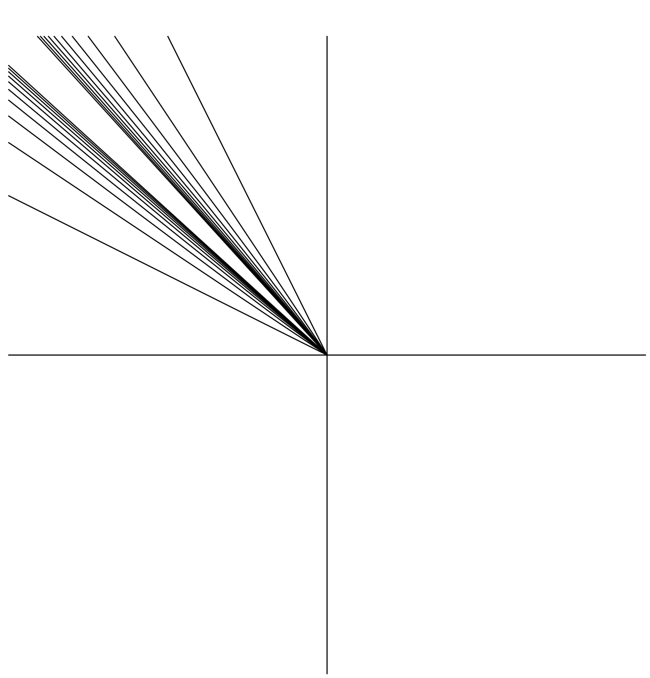
The affine groups $\tilde{A}_2$ and $\tilde{B}_2$

Lots of numerology waiting to be done!

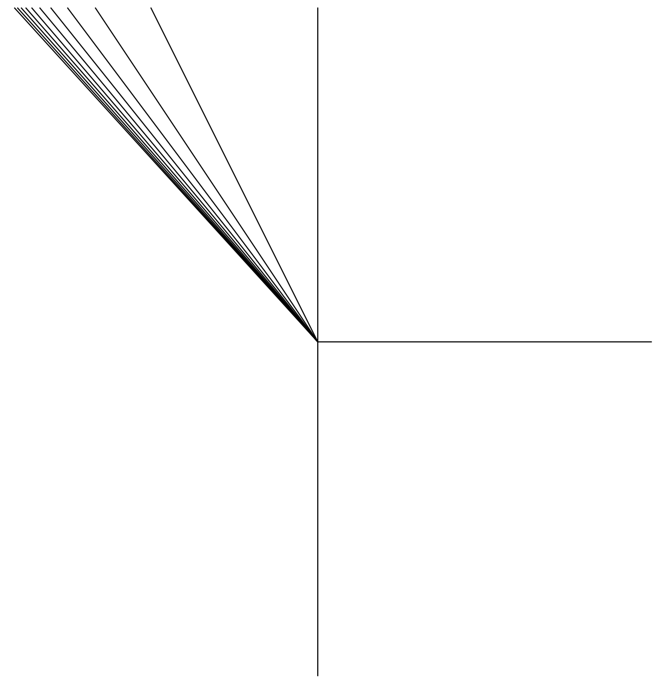


The hyperbolic group $(5, 4, 2)$

Compared to the cluster and quiver theoretic definitions, this misses a big part of the picture.



The g-complex for $\tilde{A}_1$



The $\tilde{A}_1$-Cambrian fan

We conjecture that we are seeing those cones in the g-complex that meet the interior of the Tits cone.

Problem 1: How do we escape the Tits cone?

Vague idea – we need some sort of larger lattice that includes W but also has regions for other points in V^*. Any ideas?

Coxeter elements correspond to acyclic orientations of diagrams. But, from a cluster algebra perspective, we should consider all orientations of Dynkin diagrams.

Problem 2: How do we deal with cyclic orientations?

Tenative experimentation indicates that our pattern avoidance criteria should still work, but many details need to be filled in.