Note:

pgs 1 → 6 notes from the notetaker

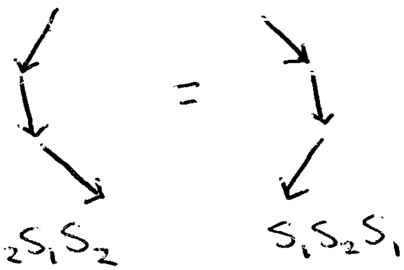
pgs 7 → 12 handwritten notes of Arun Ram

3/19/08 - 9:30AM - Arun Ram - "Path Models"

Joint work with M. Yip on Arxiv 0803.1146

$W_0 = \{\text{alcoves in } 0 \text{ hexagon}\}$

See figure on next page.



$\mathfrak{h}^*_{\mathbb{Z}} = \{\text{hexagons}\}$

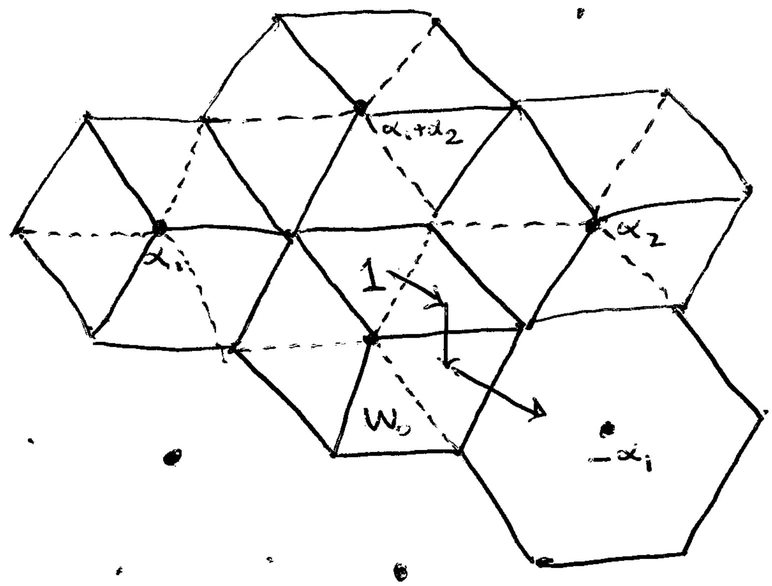
$\mathfrak{h}_{\mathbb{Z}} : \langle \mu, \lambda^\vee \rangle = \mu(\lambda^\vee)$, if $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, $\lambda \in \mathfrak{h}_{\mathbb{Z}}$

Vector spaces	Notation	Basis
Hecke algebra	H_0	$\{T_w\}$
Double Affine Hecke Alg (DAHA)	$\tilde{H}$	$\{X^\mu T_w y^{\lambda^\vee}\}$
Polynomial Representation	$\tilde{H}\mathbb{1}$	$\{X^\mu \mathbb{1}\}$
Symmetric Polynomials	$\mathbb{1}_0 \tilde{H} \mathbb{1}$	$\{m_\mu \mathbb{1}\}$

$\mu \in \mathfrak{h}^*_{\mathbb{Z}}$ $\quad w \in W_0$ $\quad \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$

$\mathfrak{h}^*_{\mathbb{Z}}$

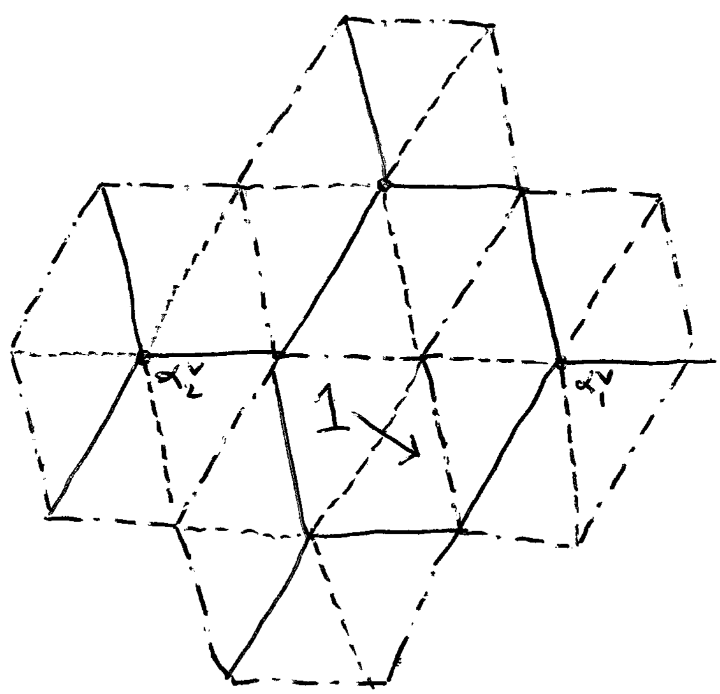
1 = Blue - - - -
2 = Red ———



$$T_{w_0} = T_1 T_2 T_1$$

$\mathfrak{h}_{\mathbb{Z}}$

1 = Blue - - - -
2 = Red ———
3 = Yellow -·-·-



<u>Relations</u> $$m_\mu = \sum_{\gamma \in W_0\mu} X^\gamma$$ monomial symmetric fcn

$$T_i^2 = (t^{1/2} - t^{-1/2})\, T_i + 1 \qquad T_i T_j T_i = T_j T_i T_j$$

$$X^\mu X^\nu = X^{\mu+\nu} \qquad y^{\lambda^\vee} y^{\sigma^\vee} = y^{\lambda^\vee + \sigma^\vee}$$

$T_w = T_{i_1} \cdots T_{i_\ell}$ if $w = s_{i_1} \cdots s_{i_\ell}$ is reduced.

$T_i \mathbb{1} = t^{1/2} \mathbb{1}$ & $y^{\lambda^\vee} \mathbb{1} = t^{\frac{1}{2}(\#\text{ of hyperplanes betw } \lambda^\vee \,\&\, 0)} \mathbb{1}$

$\mathbb{1}_0 T_i = t^{1/2} \mathbb{1}_0$ $\qquad y^{\lambda^\vee} X^\mu = q^{-\langle \mu, \lambda^\vee \rangle} X^\mu y^{\lambda^\vee}$ + lower terms

$q^c = t$

<u>Intertwiners</u> $\tau_i^\vee y^{\lambda^\vee} = y^{s_i \lambda^\vee} \tau_i^\vee$ & $\tau_i X^\mu = X^{s_i \mu} \tau_i$

where $\tau_i^\vee = T_i + \dfrac{t^{-1/2} - t^{1/2}}{1 - y^{-\alpha_i^\vee}}$ $\qquad \tau_i = T_i + \dfrac{t^{1/2} - t^{-1/2}}{1 - X^{\alpha_i}}$

Macdonald Polynomials

Fix $\mu \in \mathfrak{h}_\mathbb{Z}^*$ $\quad \vec{\mu} = s_{i_1} \cdots s_{i_\ell}$ a minimal length walk to μ hexagon.

$\tau_\mu = \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee$
intertwiner

$E_\mu = \tau_\mu \mathbb{1}$
non-sym. Macdonald polynomial

$P_\mu = \mathbb{1}_0 \tau_\mu \mathbb{1}$
Symm. Macdonald poly.

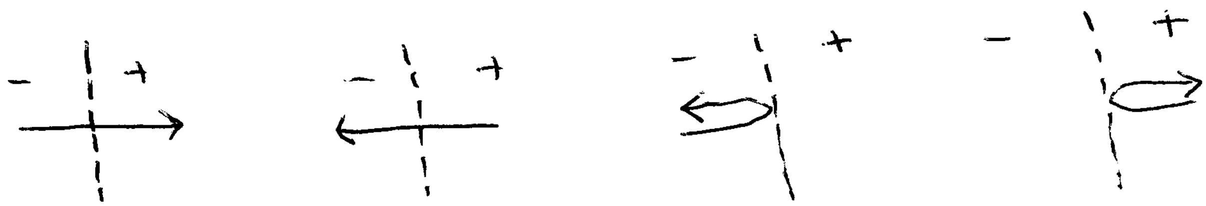
$P_\mu(0,t) = P_\mu|_{q=0}$
Mac Spherical Fcn = Hall-Littlewood Poly

$P_\mu(0,0) = s_\mu$
Weyl Char = Schur fcn

$j = \text{blue} = ----$

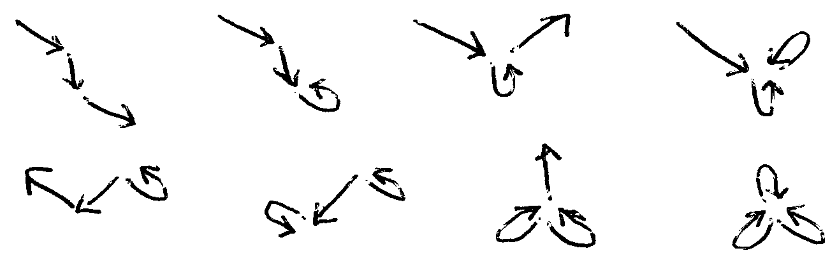
$j = 1, 2, 0$

A <u>step of type</u> j is



$$B(\vec{\mu}) = \left\{ \begin{array}{l} \text{alcove walks of type } (i_1, \ldots, i_\ell) \\ \text{beginning at } 1 \end{array} \right.$$

eg. $\mu = -\alpha_1$ $\quad \vec{\mu} = s_1 s_2 s_0$ $\quad \tau_\mu = \tau_1^\vee \tau_2^\vee \tau_0^\vee$

$B(\vec{\mu}) = \{$  $\}$

<u>Theorem</u> (Ram-Yip)

$$T_\mu = \sum_{p \in B(\vec{\mu})} X^{wt(p)} T_{\varphi(p)} F(p)$$ where end(p) is

$\varphi(p)$ alcove of $wt(p)$ hexagon.

$$F(p) = \prod_{\text{pos. fold}} \frac{t^{-1/2}(1-t)}{1 - y^{-\beta_f^\vee}} \prod_{\substack{\text{neg.} \\ \text{fold}}} \frac{t^{-1/2}(1-t)\, y^{-\beta_f^\vee}}{1 - y^{-\beta_f^\vee}}$$

This gives us formulas for $E_\mu, P_\mu, P_\mu(0,t), s_\mu$.

Formula for $P_\mu(0,t)$ is Schwer's formula.

Formula for $P_\mu(0,0)$ is Littelmann's formula.

<u>For E_μ</u>:

$$F(p)\,\mathbb{1} = \prod_{\substack{\text{pos.}\\ \text{fold}}} \frac{t^{-1/2}(1-t)}{1-q^{l(f)+1}t^{a(f)+1}} \prod_{\substack{\text{neg}\\ \text{fold}}} \frac{t^{-1/2}(1-t)\,q^{l(f)+1}t^{a(f)+1}}{1-q^{l(f)+1}t^{a(f)+1}}$$

$a(f)$ = arm length $\qquad l(f)$ = leg length

This is the form of terms in HHL formula for E_μ in type GL_n. Our formula has <u>many</u> more terms & <u>many</u> more denoms.

<u>Integral form in type GL_n</u>

$$\mathcal{E}_\mu = \Big(\prod_{\substack{\text{some}\\ \text{fold}}} 1-q^{l(f)+1}t^{a(f)+1}\Big) E_\mu$$

Our formula has too many denoms.

<u>Geometry</u> in $P_\mu(0,t)$

G_0 = Chevalley group $\qquad SL_3(\mathbb{C})$

G = Loop group $\qquad SL_3(\mathbb{C}((t)))$

$\mathbb{C}((t)) = \{ a_{-l}t^{-l} + a_{-l+1}t^{-l+1} + \cdots \mid a_i \in \mathbb{C},\ l \in \mathbb{Z} \}$

$\mathbb{C}[[t]] = \{ a_0 t^0 + a_1 t + \cdots \}$

K = "maximal compact" $\qquad SL_3(\mathbb{C}[[t]])$

$$\begin{array}{ccccc}
 & G & & & \\
 & \cup\!\text{l} & & & \\
 & K & \xrightarrow{t=0} & G_0 & \\
 & \cup\!\text{l} & & \cup\!\text{l} & \\
\text{Iwahori Subgrp} \rightarrow & I & \longrightarrow & B_0 & = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \text{ Borel}
\end{array}$$

G/K = loop Grassmannian $\qquad G/I$ = affine flag variety

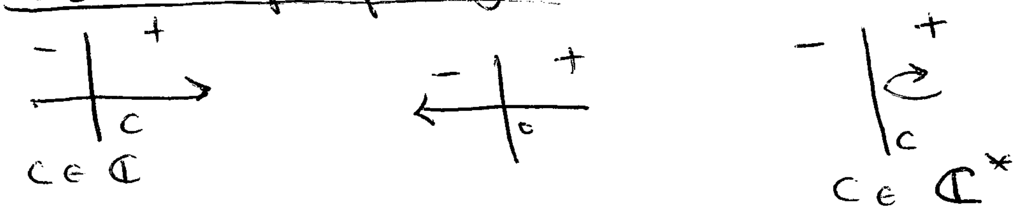
$$U^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{C}((t)) \right\}$$

$$G = \bigsqcup_{v \in W} U^- v I \qquad G = \bigsqcup_{w \in W} I w I$$

where $W = W_0 \ltimes \mathfrak{h}_{\mathbb{Z}}$.

A <u>MV - intersection</u> is $IwI \cap U^- v I$ (in G/I)

A <u>labeled step of type</u> j is



Let $w \in W$. Let $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ a reduced word.

$$\mathbb{B}(\vec{w})_v = \left\{ \begin{array}{l} \text{labeled alcove walks of} \\ \text{type } i_1, \ldots, i_\ell \text{ which end in } v \end{array} \right\}$$

<u>Theorem</u> (Parkinson R - Schwer) following Gaussant - Littelmann

$$IwI \cap U^- v I \xleftrightarrow{1-1} \mathbb{B}(\vec{w})_v$$

<u>Summary of Talk</u> (in response to Fomin's Question)

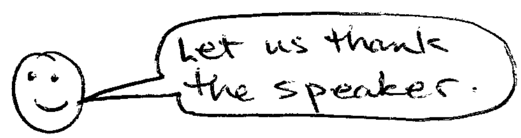
DAHA $\qquad$ AHA $\cong C_c(K \backslash G/K)$

$$\mathbb{C}[P]^W = Z(H) \xrightarrow{\sim} \mathbb{1}_0 H \mathbb{1}_0$$

$$P_\mu = \mathbb{1}_0 T_\mu \mathbb{1} \qquad P_\mu(0,t) \longleftrightarrow \mathbb{1}_0 X^\mu \mathbb{1}_0$$

Setting $q=0$, the DAHA becomes AHA.

End of Lecture.

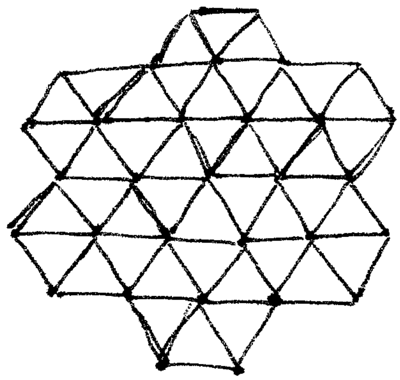


MSRI Conference, March 19, 2008, "Path Models".

DATA and DAHA

$W_0 = \left\{ \begin{matrix} \text{alcoves in} \\ \text{0-hexagon} \end{matrix} \right\}$ $\qquad \mathfrak{h}^*_{\mathbb{Z}} = \{\text{hexagons}\}$

$\mathfrak{h}_{\mathbb{Z}}: \quad \langle \mu, \lambda^\vee \rangle = \mu(\lambda^\vee), \quad \text{for } \mu \in \mathfrak{h}^*_{\mathbb{Z}}, \ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}.$



Vector Space	Notation	$\mathbb{C}[q^{\pm 1}]$-basis.
Hecke algebra	H_0	$\{T_w\}$
Double affine Hecke algebra	$\tilde{H}$	$\{X^\mu T_w Y^{\lambda^\vee}\}$
Polynomial representation	$\tilde{H}\mathbb{1}$	$\{X^\mu \mathbb{1}\}$
Symmetric polynomials	$\mathbb{1}_0 \tilde{H} \mathbb{1}$	$\{m_\mu \mathbb{1}\}$

$\mu \in \mathfrak{h}^*_{\mathbb{Z}}, \qquad w \in W_0, \qquad \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$

Relations

$$m_\mu \mathbb{1} = \sum_{\gamma \in W_0\lambda} X^\gamma \mathbb{1}, \qquad \mathbb{1}_0 T_i = t^{\frac{1}{2}} \mathbb{1}_0$$

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1, \qquad T_i T_j T_i = T_j T_i T_j, \qquad t = q^c$$

$$T_w = T_{i_1} \cdots T_{i_\ell}, \quad \text{if} \quad w = s_{i_1} \cdots s_{i_\ell} \quad \text{is reduced.}$$

$$X^\mu X^\nu = X^{\mu+\nu}, \qquad y^{\lambda^\vee} y^{\nu^\vee} = y^{\lambda^\vee + \nu^\vee}$$

$$T_i \mathbb{1} = t^{\frac{1}{2}} \mathbb{1}, \qquad y^{\lambda^\vee} \mathbb{1} = t^{\frac{1}{2}(\#\text{ of hyp. between } \lambda^\vee \text{ and } 0^\vee)} \mathbb{1}$$

$$y^{\lambda^\vee} X^\mu = q^{-\langle \lambda^\vee, \mu \rangle} X^\mu y^{\lambda^\vee} + \text{lower terms}$$

Intertwiners

$$\tau_i^\vee y^{\lambda^\vee} = y^{s_i \lambda^\vee} \tau_i^\vee \quad \text{and} \quad \tau_i X^\mu = X^{s_i \mu} \tau_i$$

where

$$\tau_i^\vee = T_i + \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})}{1 - y^{-\alpha_i^\vee}} \qquad \text{and}$$

$$\tau_i = T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - X^{\alpha_i}}.$$

Macdonald polynomials (MP)

Fix μ ~~$= s_{i_1} \cdots s_{i_\ell}$~~ $\in \mathfrak{h}^*_{\mathbb{Z}}$ and $\vec{\mu} = s_{i_1} \cdots s_{i_\ell}$

a minimal length walk to the μ-hexagon.

$$\tau_\mu = \tau^\vee_{i_1} \cdots \tau^\vee_{i_\ell}, \qquad E_\mu = \tau_\mu \mathbf{1}, \qquad P_\mu = \mathbf{1}_0 \tau_\mu \mathbf{1}$$

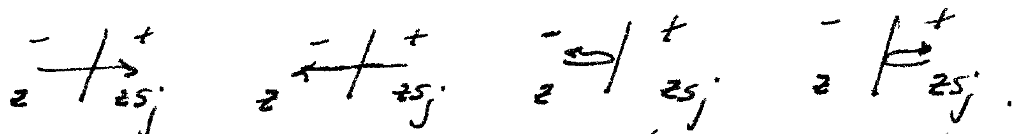
μ-intertwiner   nonsymmetric MP   symmetric MP

$P_\mu(0,t)$ $P_\mu(0,0)$

Macdonald spherical fcn   Weyl character

Hall-Littlewood poly.   Schur function

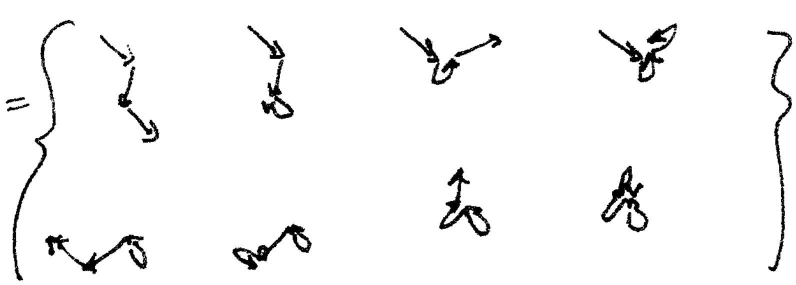
A *step of type* j is



negative fold   positive fold

$$B(\vec{\mu}) = \left\{ \begin{array}{l} \text{alcove walks of type } i_1, \ldots, i_\ell \\ \text{beginning at } \mathbf{1} \end{array} \right\}$$

Example: $\mu = \alpha_2$ $\vec{\mu} = s_1 s_2 s_0 =$ 

$B(\vec{\mu}) =$ 

<u>Theorem</u> (Ram-Yip)

$$T_\mu = \sum_{p \in B(\vec{\mu})} X^{wt(p)} T_{\varphi(p)} F(p), \quad \text{where}$$

$end(p)$ is $\varphi(p)$ alcove of $wt(p)$ hexagon.

$$F(p) = \prod_{\substack{\text{positive}\\ \text{folds } f}} \frac{t^{-\frac{1}{2}}(1-t)}{1-y^{-\beta_f^\vee}} \prod_{\substack{\text{negative}\\ \text{folds } f}} \frac{t^{-\frac{1}{2}}(1-t)\, y^{-\beta_f^\vee}}{1-y^{-\beta_f^\vee}}$$

<u>Remarks</u>: We get formulas for $E_\mu, P_\mu, P_\mu(0,t), P_\mu(0,0)$:

$$F(p)\mathbb{1} = \prod_{\substack{\text{pos}\\ \text{folds}}} \frac{t^{\frac{1}{2}}(1-t)}{1-q^{l(f)+1}t^{a(f)+1}} \prod_{\substack{\text{neg}\\ \text{folds}}} \frac{t^{\frac{1}{2}}(1-t)\, q^{l(f)+1}t^{a(f)+1}}{1-q^{l(f)+1}t^{a(f)+1}} \qquad (*)$$

$a(f)$ = arm length, $l(f)$ = leg length

Formula for $P_\mu(0,t)$ is <u>Schwer's formula</u>

Formula for $P_\mu(0,0)$ is <u>Littelmann's formula</u> (Gaussent-Littelmann).

$(*)$ has the same form as in the

HHL formula for E_μ in type GL_n.

HHL formula has

<u>many</u> fewer terms and <u>many</u> fewer denominators.

<u>Integral forms</u>: In type GL_n:

$$\mathcal{E}_\mu = \prod_{\substack{\text{some}\\ \text{folds}}} \left(1 - q^{\ell(h)+1} t^{a(h)+1}\right) E_\mu \quad \text{has no denominators.}$$

Our formula has <u>too</u> many denominators.

Integral forms in other types?

<u>Geometry of $\mathcal{P}_\mu(0,t)$</u> (positively folded).

G_0 = Chevalley group $\qquad\qquad SL_3(\mathbb{C})$

G = Loop group $\qquad\qquad SL_3(\mathbb{C}((t)))$

$$\mathbb{C}((t)) = \{ a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots \mid \ell \in \mathbb{Z},\ a_i \in \mathbb{C} \}$$

K = "maximal compact" $\qquad\qquad SL_3(\mathbb{C}[[t]])$

$$U^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{C}((t)) \right\}$$

$$\begin{tikzcd}
 & G & \\
 & K \arrow[u, phantom, "\cup|"] \arrow[r, "t=0"] & G_0 \\
\text{Iwahori subgroup} & I \arrow[u, phantom, "\cup|"] \arrow[r] & B_0 \arrow[u, phantom, "\cup|"] = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right\} \quad \text{Borel subgroup}
\end{tikzcd}$$

G/K is the <u>loop Grassmanian</u>

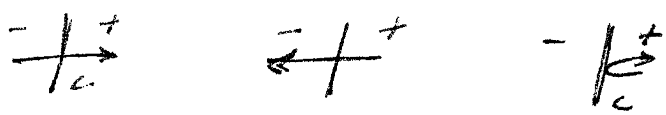
G/I is the <u>affine flag variety</u>

$$G = \bigsqcup_{v \in W} U^- v I \quad \text{and} \quad G = \bigsqcup_{w \in W} I w I$$

where $W = W_0 \ltimes \mathbb{Z}^{n}$. An <u>MV intersection</u> is

$$U^- v I \cap I w I \qquad \text{in } G/I$$

Let $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ be a reduced word. A <u>labeled step of type j</u> is



$c \in \mathbb{C}$ $\qquad\qquad\qquad\qquad$ $c \in \mathbb{C}^\times$

$$B(\vec{w})_v = \left\{ \begin{array}{l} \text{labeled alcove walks of type } i_1, \ldots, i_\ell \\ \text{ending at } v \end{array} \right\}$$

<u>Theorem</u> (Parkinson-R-Schwer) small strengthening of Gaussent-Littelmann

$$I w I \cap U^- v I \xleftrightarrow{1-1} B(\vec{w})_v$$