k-Schur functions via multidegrees of matrix affine Schubert varieties

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Why k-Schur functions?

- [Lapointe,Lascoux,Morse] "Positively triangular" with Macdonald polynomials; refinement of Macdonald positivity
- [Lam] Schubert basis of $H_*(\mathrm{Gr}_{SL_n})$.
- [Peterson] [Lam,S.] The Schubert structure constants for $H_*(\mathrm{Gr}_G)$ are the same as those for $QH^*(G/B)$.

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- [Knutson-Miller] Geometric explanation of monomial expansion of Schur polynomial (Schubert varieties in the Grassmannian)
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Schur functions via Pieri

$$\sum_{r\in\mathbb{Z}} h_r(x)t^r = \prod_{i\geq 1}(1 - x_i t)^{-1}$$

$$h_\mu = h_{\mu_1} h_{\mu_2} \cdots$$

$$\Lambda = \mathbb{Z}[h_1, h_2, \ldots] = \bigoplus_{\mu\in\mathbb{Y}} \mathbb{Z} h_\mu$$

There is a unique $\mathbb{Z}$-basis $\{s_\lambda \mid \lambda \in \mathbb{Y}\}$ of Λ such that

$$h_r s_\mu = \sum_\lambda s_\lambda$$

λ/μ is a horizontal strip of size r. ($h_r = s_{(r)}$)

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$$h_2 s_{(3,2)} = s_{(3,2,2)} + s_{(3,3,1)} + s_{(4,2,1)} + s_{(4,3)} + s_{(5,2)}$$

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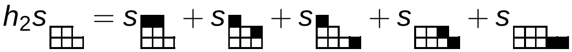

Take $s_\varnothing = 1$ and multiply by h_{μ_1}, then h_{μ_2}, ...

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$K_{\lambda\mu}$: number of tableaux (sequences of horizontal strips) from $\varnothing$ to λ of weight μ.

$K_{\lambda\mu}$ is invariant under permuting μ.

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2		
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shape $(3, 1, 1)$ weight $(2, 2, 1)$

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$$s_\lambda = \sum_{\beta \in \mathbb{Z}_{\geq 0}^{\infty}} K_{\lambda\beta} x^\beta$$
$$= \sum_{\mathrm{shape}(T)=\lambda} x^T$$

$$s_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}}(x_1, x_2) = x^{\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 1 \\ \hline\end{array}} + x^{\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 2 \\ \hline\end{array}}$$

$$= x_1^2 x_2 + x_1 x_2^2$$

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Grassmannian

$\mathrm{Gr}(k,n)$: k-dimensional subspaces of $\mathbb{C}^n$
GL_k acts freely on $M^o_{k\times n}$ (full rank matrices)

$$\begin{aligned}\pi : M^o_{k\times n} &\twoheadrightarrow \mathrm{Gr}(k,n)\\ A &\mapsto \mathrm{rowspace}(A)\\ GL_k\backslash M^o_{k\times n} &\cong \mathrm{Gr}(k,n)\end{aligned}$$

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Schubert cells in $\mathrm{Gr}(k,n)$

$A \in M^o_{k\times n}$ has row-echlon form with pivots in some set of columns $I = \{i_1 < i_2 < \cdots < i_k\} \in \binom{[n]}{k}$ $\lambda \subset [k] \times [n-k]$

Example: $n = 6$, $k = 2$, $I = \{3,5\}$, $\lambda(I) = \yng(2,3)$

$$X^o_{\yng(2,3)} = \left\{ \pi \begin{pmatrix} 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{pmatrix} \right\}$$ Schubert cell

$${X_{\yng(2,3)}}^o \cong \mathbb{C}^3 \qquad \operatorname{codim} X_{\yng(2,3)} = 5$$

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Schubert basis of $H^*(\mathrm{Gr}(k,n))$

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$$H^*(\mathrm{Gr}(k,n)) \cong \bigoplus_{\lambda \subset [k]\times[n-k]} \mathbb{Z}[X_\lambda]^*$$

$$X_\lambda = \overline{X_\lambda^o} \qquad \text{Schubert variety}$$

$$H^*(\mathrm{Gr}(k,n)) \cong \mathbb{Z}[x_1,\ldots,x_k]^{S_k} / \bigoplus_{\lambda \not\subset [k]\times[n-k]} \mathbb{Z}s_\lambda(x_1,\ldots,x_k)$$

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Matrix Schubert variety Y_λ

$$M_{k\times n} \supset Y_\lambda := \overline{\pi^{-1}(X_\lambda)}$$

$$Y_{(3,2)} = \overline{GL_2 \cdot \begin{pmatrix} 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}}$$

$A \in M^o_{k\times n}$

$r_i(A)$ rank of first i columns of A

i	0	1	2	3	4	5	6
$r_i(A)$	0	0	0	1	1	2	2

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Multidegree

$T \cong (\mathbb{C}^*)^m$
M fin. dim. vector space with T-action
$X \subset M$: T-stable subscheme
$\mathrm{mdeg}_M(X) \in H_T^*(M) \cong \mathbb{Z}[x_1, \ldots, x_m]$ multidegree

- If $X = M = 0$ then $\mathrm{mdeg}_M(X) = 1$.
- If X is reducible then $\mathrm{mdeg}_M X = \sum_i \mathrm{mult}_{X_i}(X)\,\mathrm{mdeg}_M X_i$

T-stable hyperplane $H \subset M$

- If X is irr. and $X \not\subset H$ then $\mathrm{mdeg}_M(X) = \mathrm{mdeg}_H(X \cap H)$.
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Multidegree computation

$\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & \mathbb{C}^* \end{pmatrix} \subset GL_2$ acts on $M = M_{2\times 4} \supset Y_{\square\square}$ $\quad I = \{2, 4\}$

$$\begin{matrix} x_1 \\ x_2 \end{matrix} \quad \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	0	1	2	3	4
$r_i(A)$	0	0	1	1	2

$V(z_{11}, z_{21}, z_{12}z_{23} - z_{13}z_{22})$
Cut with $z_{11} = 0$: (contained) hold $x_1 \cdot$
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$\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & \mathbb{C}^* \end{pmatrix} \subset GL_2$ acts on $M = M_{2\times 4} \supset Y_{\square\square}$ $\quad I = \{2,4\}$

$$\begin{matrix} x_1 \\ x_2 \end{matrix} \qquad \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	0	1	2	3	4
$r_i(A)$	0	0	1	1	2

$V(z_{11}, z_{21}, z_{12}z_{23} - z_{13}z_{22})$
Cut with $z_{11} = 0$: (contained) hold $x_1\cdot$
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Gröbner degeneration

Multidegree calculation all at once:
Term order $z_{11}, z_{21}, \ldots, z_{12}, z_{22}, \ldots$.
Earlier terms are cheaper
Gröbner basis: $\{z_{11}, z_{21}, z_{12}z_{23} - z_{13}z_{22}\}$
Initial terms: $z_{11}, z_{21}, z_{13}z_{22}$
components: $V(z_{11}, z_{21}, z_{13}) \cup V(z_{11}, z_{21}, z_{22})$
Punchcards:

$x_1^2 x_2 \quad x_1 x_2^2$

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Why did it work?

$$M_{k\times n} \longleftarrow\!\!\supset M^o_{k\times n} \xrightarrow{\pi} \mathrm{Gr}(k,n)$$

$$H^*_{GL_k}(M_{k\times n}) \twoheadrightarrow H^*_{GL_k}(M^o_{k\times n}) = H^*(\mathrm{Gr}(k,n))$$

$$\mathbb{Z}[x_1,\ldots,x_k]^{S_k} \qquad \mathbb{Z}[x_1,\ldots,x_k]^{S_k} / \bigoplus_{\lambda\not\subset[k]\times[n-k]} \mathbb{Z}s_\lambda$$

$$[Y_\lambda]^*_{GL_k} \qquad [\pi^{-1}(X_\lambda)]^*_{GL_k} \qquad [X_\lambda]^*$$

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Theorem [Knutson,Miller]

$$\mathrm{mdeg}_{M_{k\times n}}(Y_\lambda) = s_\lambda(x_1,\ldots,x_k).$$

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Infinite Grassmannian Gr_∞

$$\mathbb{C}^\infty = \{(\dots, a_{-1}, a_0, a_1, \dots) \mid a_i \in \mathbb{C},\ a_i = 0 \text{ for } i << 0\}$$

standard “basis" $\{e_i \mid i \in \mathbb{Z}\}$ $E_m \subset \mathbb{C}^\infty$: “span" $\{e_m, e_{m+1}, \dots\}$

$$\mathrm{Gr}_\infty := \bigcup_{m \le 0 \le M} F_{m,M}$$

$$F_{m,M} := \{\text{subspaces } V \subset \mathbb{C}^\infty \mid E_m \supset V \supset E_M\}$$

$$\cong \bigsqcup_{i=0}^{M-m} \mathrm{Gr}(i, E_m/E_M) \qquad \text{connected comps.}$$

$$V \mapsto V/E_M.$$

i	$< m$	m	$m+1$	$\cdots$	$M-1$	$\geq M$
a_i	0	?	?	$\cdots$	?	$*$

Gr^d_∞: component of base point $E_{1-d} \in \mathrm{Gr}_\infty$

$E_1 \in \mathrm{Gr}^0_\infty$

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Schubert cell $Z_I^o \subset \mathrm{Gr}_\infty$ for each pivot set $I \subset \mathbb{Z}$.

$$\mathrm{Gr}_\infty^0 = \bigsqcup_{v(I)=0} Z_I^o$$

$$v(I) := |I \cap \mathbb{Z}_{\leq 0}| - |\mathbb{Z}_{>0} \setminus I|$$

$$H_*(\mathrm{Gr}_\infty^0) \cong \Lambda \qquad H^*(\mathrm{Gr}_\infty^0) \cong \Lambda$$

$$[Z_I]_* \mapsto s_{\lambda(I)} \qquad [Z_I]^* \mapsto s_{\lambda(I)}$$

trace edge of partition $\lambda(I)$: down steps I, right steps $\mathbb{Z} \setminus I$.

$$\mathrm{Gr}_\infty^0 \supset \mathrm{Gr}(M-1, M-m)$$

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Affine symmetric group $\widetilde{S}_n$

Generators $s_0, s_1, \ldots, s_{n-1}$; indices in $\mathbb{Z}/n\mathbb{Z}$
Relations

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_j &= s_j s_i \qquad i,j \text{ not adjacent} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \end{aligned}$$

$\widetilde{S}_n \to S_{\mathbb{Z}}$

$$w(i+n) = w(i) + n$$
$$\sum_{i=1}^{n} w(i) = \sum_{i=1}^{n} i$$

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$n = 3$, $w = s_0 s_1 s_2 s_1 s_0$ has window $[-3, 2, 7]$

$$[0|1,2,3|4] \xrightarrow{s_0}$$
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$\widetilde{S}_n^0$ min length coset reps in $\widetilde{S}_n / S_n$: increasing window

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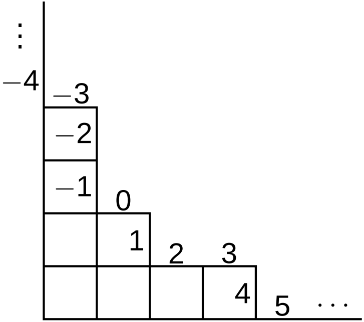
$\widetilde{S}_n^0$ min length coset reps in $\widetilde{S}_n / S_n$: increasing window

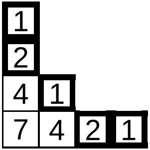
The Coset, cores, bounded partitions

$n = 3$; $w = [-3, 2, 7]$ window of $w \in \widetilde{S}_3^0$

$$I_w := \{-3, 0, 3, 6, 9 \dots\} \cup \{2, 5, 8, \dots\} \cup \{7, 10, \dots\}$$
$$= \{-3, 0, 2, 3, 5, 6, 7, 8, \dots\}$$

Core $(4, 2, 1, 1)$, bounded partition $(2, 1, 1, 1)$





k-Schur function

$$\Lambda_{(n)} := \mathbb{Z}[h_1, \ldots, h_{n-1}] = \bigoplus_{\lambda_1 < n} \mathbb{Z} h_\lambda$$

Theorem [Lapointe,Morse] There is a unique basis $s_\lambda^{(n-1)}$ indexed by n-cores, such that

$$h_r \, s_\lambda^{(n-1)} = \sum_\mu s_\mu^{(n-1)}$$

μ an n-core, $\lambda \leq \mu$ in left weak order, μ/λ horiz. strip.

Affine Grassmannian

$$\begin{aligned}\mathrm{Gr}_{SL_n} &:= SL_n(\mathbb{C}((t)))/SL_n(\mathbb{C}[[t]]) \\ &= G_{\mathrm{af}}/P_{\mathrm{af}} \\ &= \bigsqcup_{w\in\widetilde{S}_n^0} B_{\mathrm{af}} w P_{\mathrm{af}}/P_{\mathrm{af}}\end{aligned}$$

$X_w = \overline{B_{\mathrm{af}} w P_{\mathrm{af}}/P_{\mathrm{af}}}$ Schubert variety

$$H^*(\mathrm{Gr}_{SL_n}) \cong \bigoplus_{w\in\widetilde{S}_n^0} [X_w]^*$$

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[Quillen]

$$H_*(\mathrm{Gr}_{SL_n}) \cong H_*(\Omega SU(n))$$

[Bott] There is a Hopf algebra isomorphism

$$H_*(\mathrm{Gr}_{SL_n}) \to \Lambda_{(n)}$$
$$[X_{s_{r-1}\cdots s_1 s_0}]_* \mapsto h_r \qquad 1 \leq r \leq n-1$$

[Lam] It sends Schubert classes to k-Schur functions:

$$[X_w]_* \mapsto s_\lambda^{(n-1)}$$

λ is the core for $w \in \widetilde{S}_n^0$.

Lusztig lattice model

$L \in \mathrm{Gr}_{GL_n}$ lattice: $\mathbb{C}[[t]]$-submodule of $\mathbb{C}((t))^n$ of rank n

$L \subset \mathbb{C}((t))^n \cong \mathbb{C}^\infty$

$t^j e_i \mapsto e_{i+nj}$ for $j \in \mathbb{Z}$ and $1 \le i \le n$

$\mathrm{rank}(L) = n \Rightarrow$ There is a pivot in each congruence class mod n. Pick the minimum one in each class, and sort them: $w(1) < w(2) < \cdots < w(n)$

$tL \subset L$ so pivot set I_L is stable under adding n.

$I_L = \{w(1), w(2), \ldots, w(n)\} + n\mathbb{Z}_{\ge 0}$

$$\mathrm{Gr}_{GL_n} \subset \mathrm{Gr}_\infty$$
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For $L \in \mathrm{Gr}_{SL_n}$, $\lambda(I_L)$ is an n-core.

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$L \in \mathrm{Gr}_{GL_n}$ lattice: $\mathbb{C}[[t]]$-submodule of $\mathbb{C}((t))^n$ of rank n
$L \subset \mathbb{C}((t))^n \cong \mathbb{C}^\infty$
$t^j e_i \mapsto e_{i+nj}$ for $j \in \mathbb{Z}$ and $1 \le i \le n$
$\mathrm{rank}(L) = n \Rightarrow$ There is a pivot in each congruence class mod n. Pick the minimum one in each class, and sort them:
$w(1) < w(2) < \cdots < w(n)$
$tL \subset L$ so pivot set I_L is stable under adding n.
$I_L = \{w(1), w(2), \ldots, w(n)\} + n\mathbb{Z}_{\ge 0}$

$$\mathrm{Gr}_{GL_n} \subset \mathrm{Gr}_\infty$$
$$\mathrm{Gr}_{SL_n} \subset \mathrm{Gr}^0_\infty$$

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Schubert cell X_w^o

Pivot set $\{-3, 2, 7\} + 3\mathbb{Z}_{\geq 0} = \{-3, 0, 2, 3, 5, 6, 7, \dots\}$

-3	1							
-2	a							
-1	b							
0		1						
1	c	a						
2		b	1					
3				1				
4	d	c	e	a				
5					1			
6						1		
7							1	
8								1

-3	1						
-2	a						
-1	b						
0		1					
1	c	a					
2			1				
3				1			
4	d	c'	e	a			
5					1		
6						1	
7							1
8							

$w \in \widetilde{S}_n^0$, λ associated n-core
In Gr_∞^0,

$$X_w = Z_\lambda \cap \mathrm{Gr}_{SL_n}$$

Let a, b be large enough so that $\lambda \subset [a] \times [a+b]$.

$$Z_\lambda \subset \mathrm{Gr}(a, a+b) \subset \mathrm{Gr}_\infty^0$$

$$M_{a\times(a+b)} \hookleftarrow M^o_{a\times(a+b)} \xrightarrow{\pi} \mathrm{Gr}(a, a+b)$$

Define the matrix affine Schubert variety by

$$Y_w = \overline{\pi^{-1}(X_w)}$$

Mark Shimozono - "K-Schur functions via multidegrees of Matrix Affine Schubert varieties" - 3/20/08 11AM

medium: Power point presentation

Commentary:

1) The speaker notes that this is joint work with Cory, his student. He reminds the audience to not work on some of the current work his student is doing now.

2) Table on 2^{nd} to last slide is the transpose of the intended table. (re. Schubert cell X_w° slide)

3) Kreiman Lakshmibai, Magyar, Weyman's work was cited.

End of Talk Blackboard notes:

4) def^n of matrix affine Schubert variety

$$M_{a,a+b} \supset M^\circ_{a,a+b} \xrightarrow{\pi} Gr(a, a+b)$$

$$\pi^{-1}(X_w) \qquad X_w$$

$Y_w := \pi^{-1}(X_w)$ is the matrix affine Schubert variety.

$$H^\bullet_{Gl_a}(M_{a,a+b}) \twoheadrightarrow H^\bullet_{GL_a}(M^\circ_{a,a+b}) \cong H^\bullet(Gr(a,a+b))$$

$$[Y_w]^\bullet_{Gl_a} \qquad\qquad \downarrow$$

$$H_\bullet(Gr(a,a+b)) \subset H_\bullet(Gr^\circ_\infty)$$

$$H_\bullet(Gr_{SL_n}) \qquad [X_\lambda]_\bullet$$