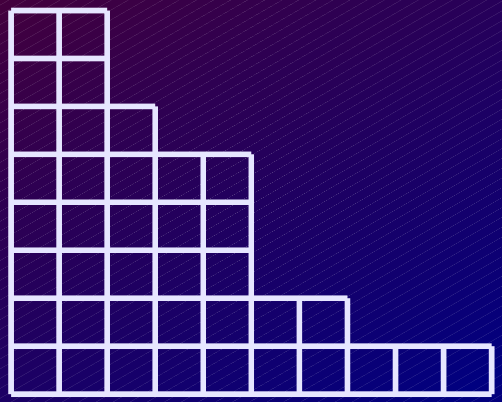
LLT Polynomials

MSRI Workshop on Topics in Combinatorial Representation Theory

Mark Haiman

joint work with

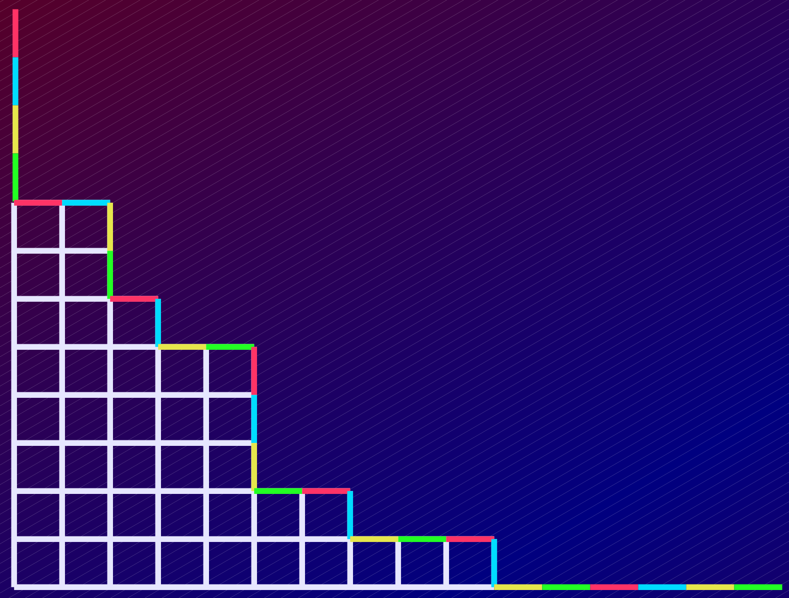
Ian Grojnowski



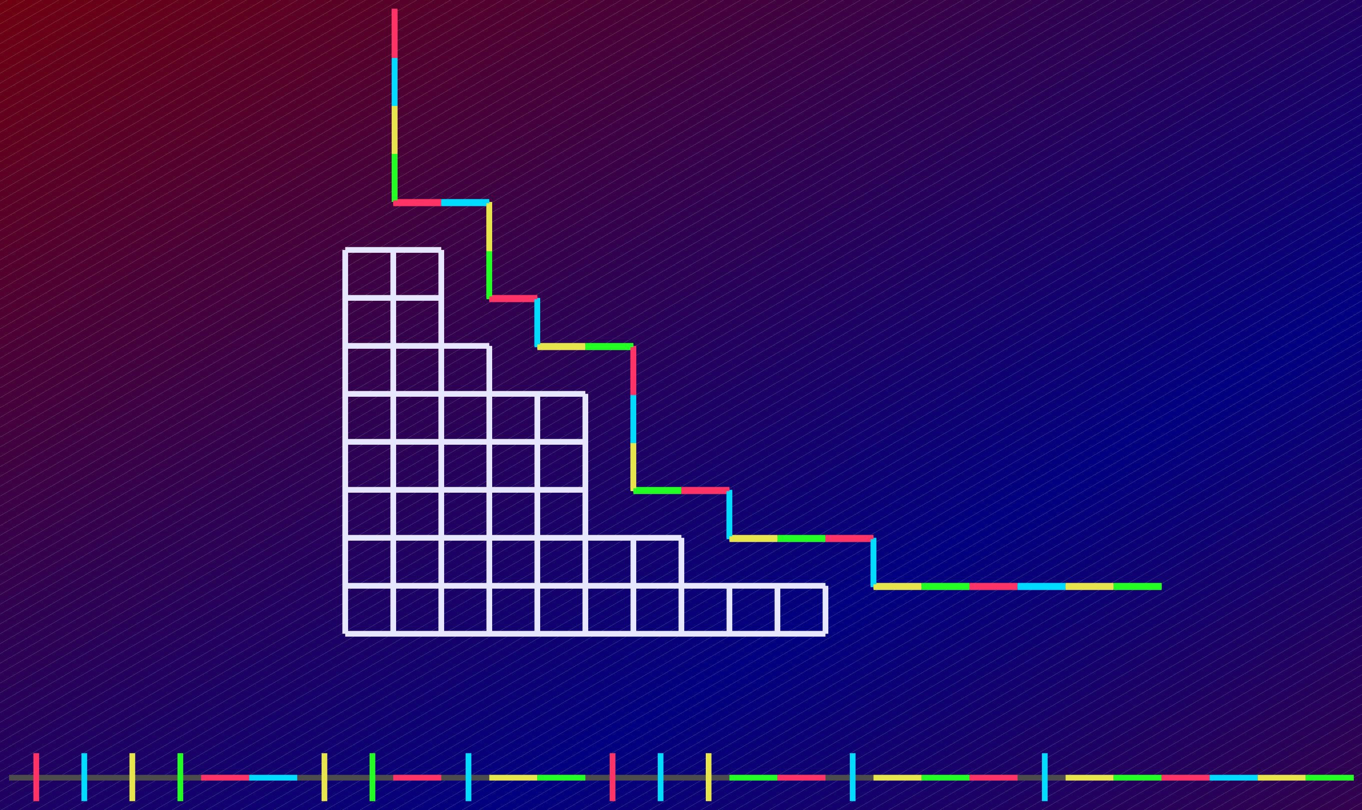
The k-core and k-quotient of a partition

(k = 4)

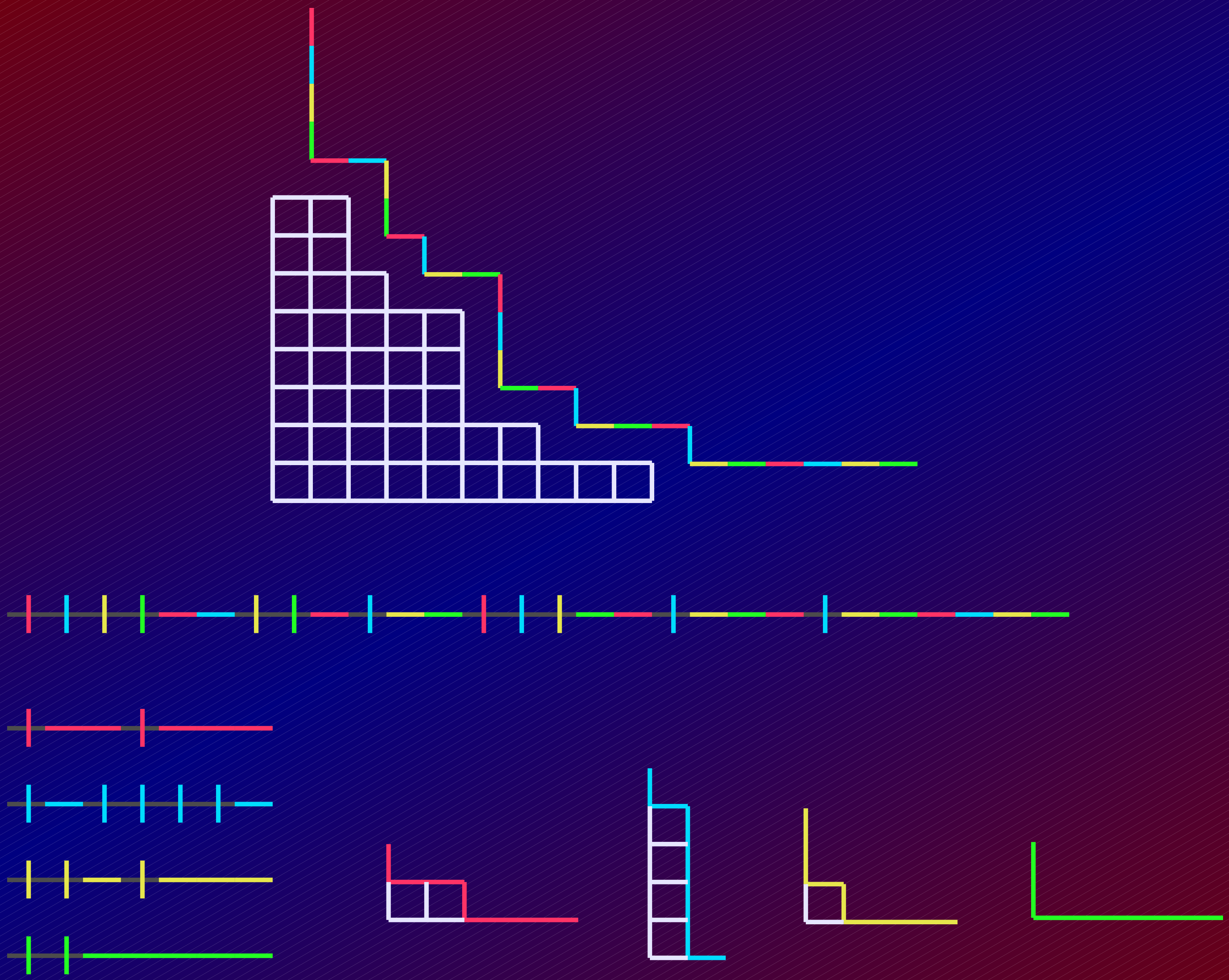
LLT Polynomials



LLT Polynomials

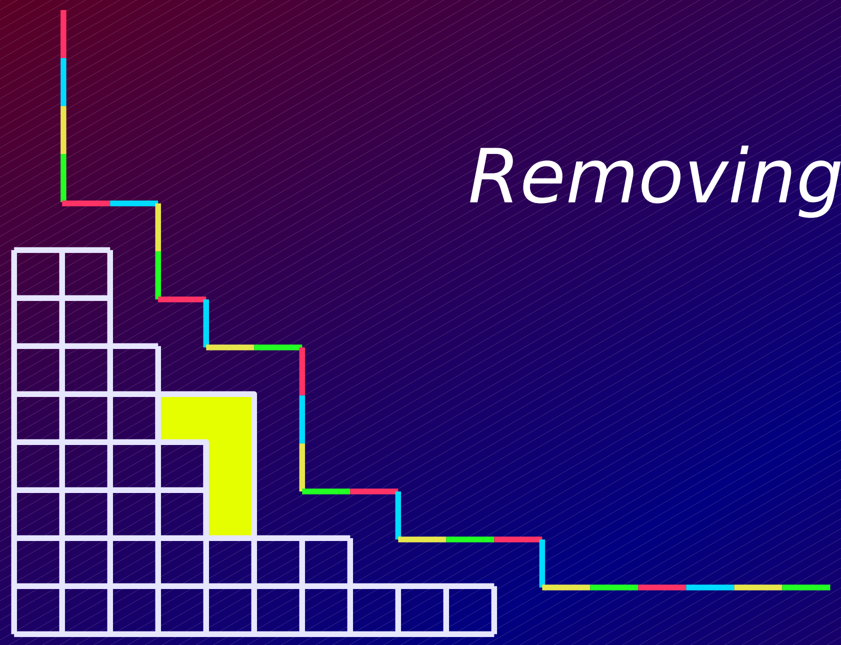


LLT Polynomials



*The **4-quotient** of our partition*

Removing *4-ribbons*...



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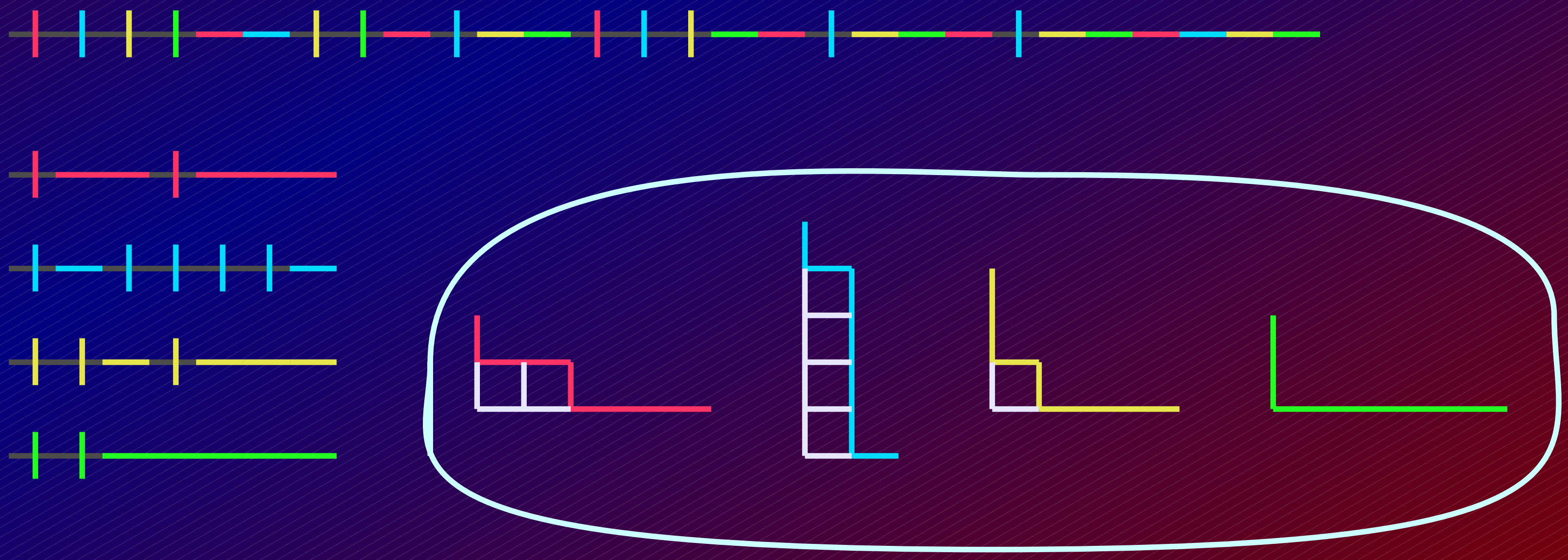

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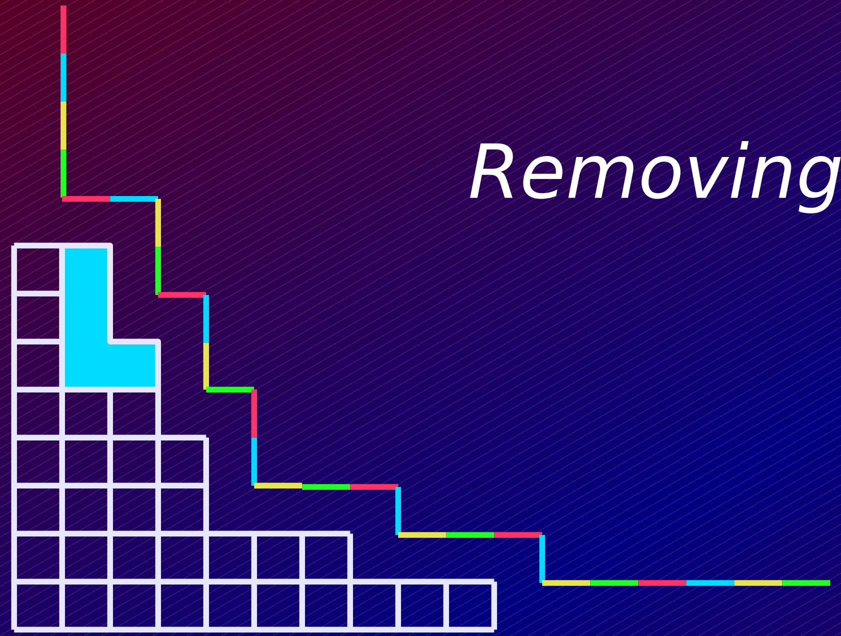

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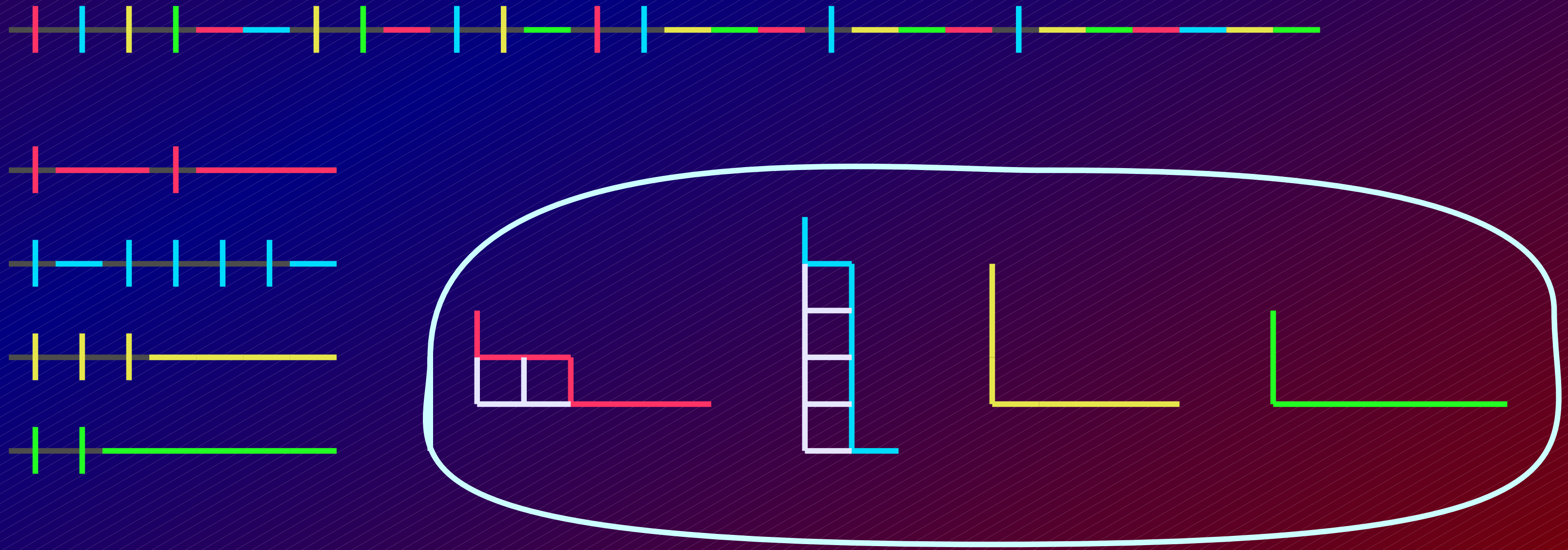

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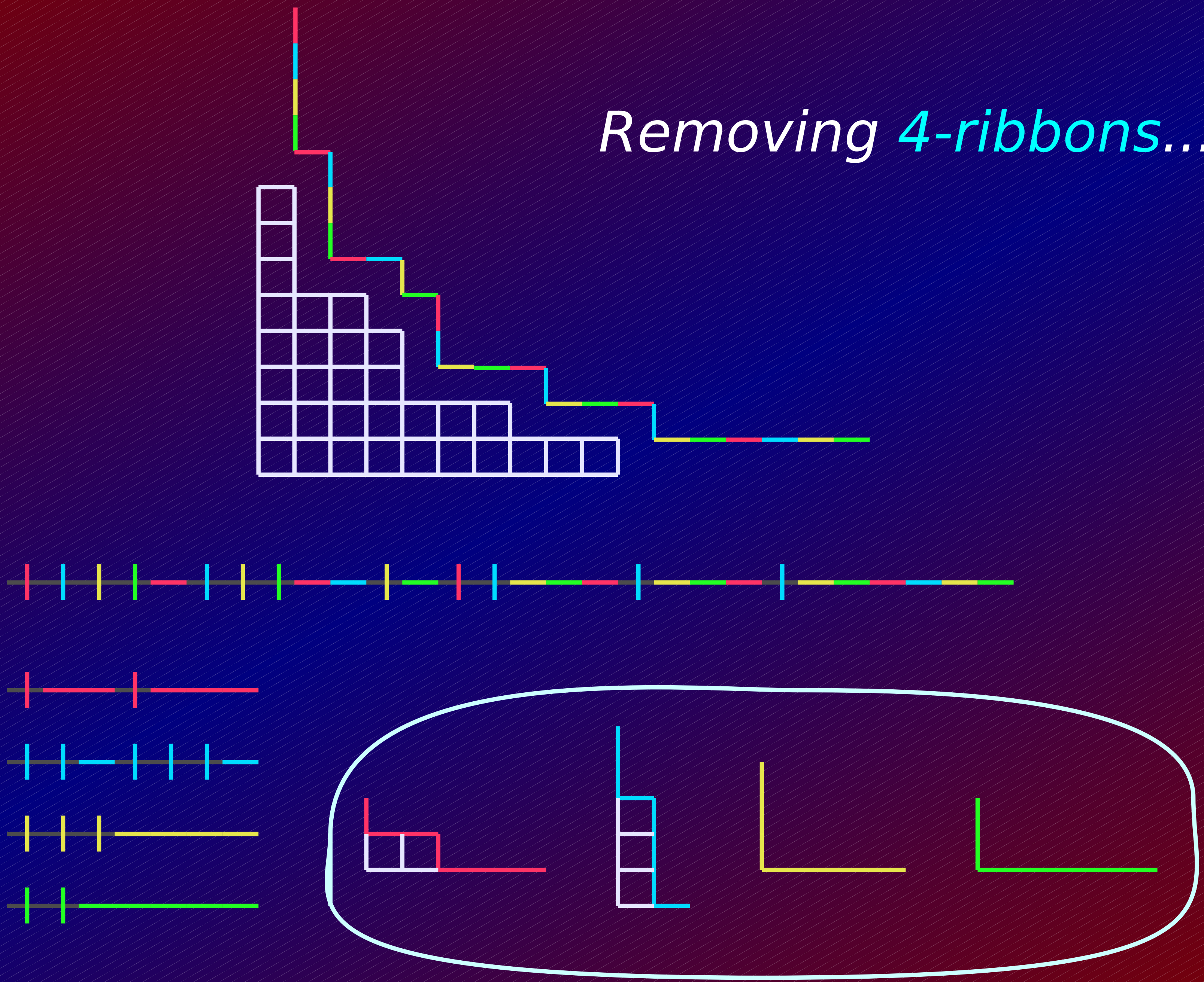

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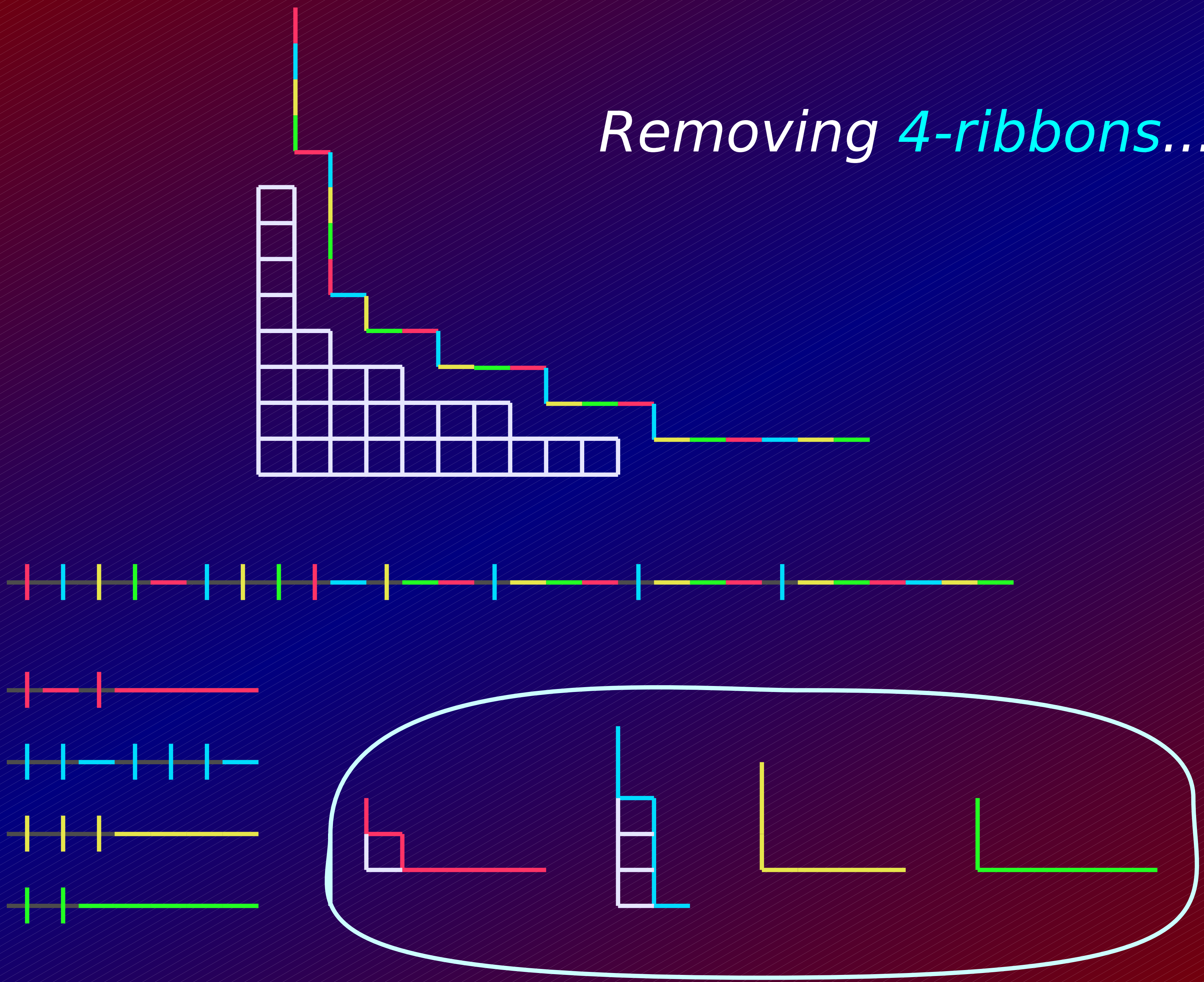

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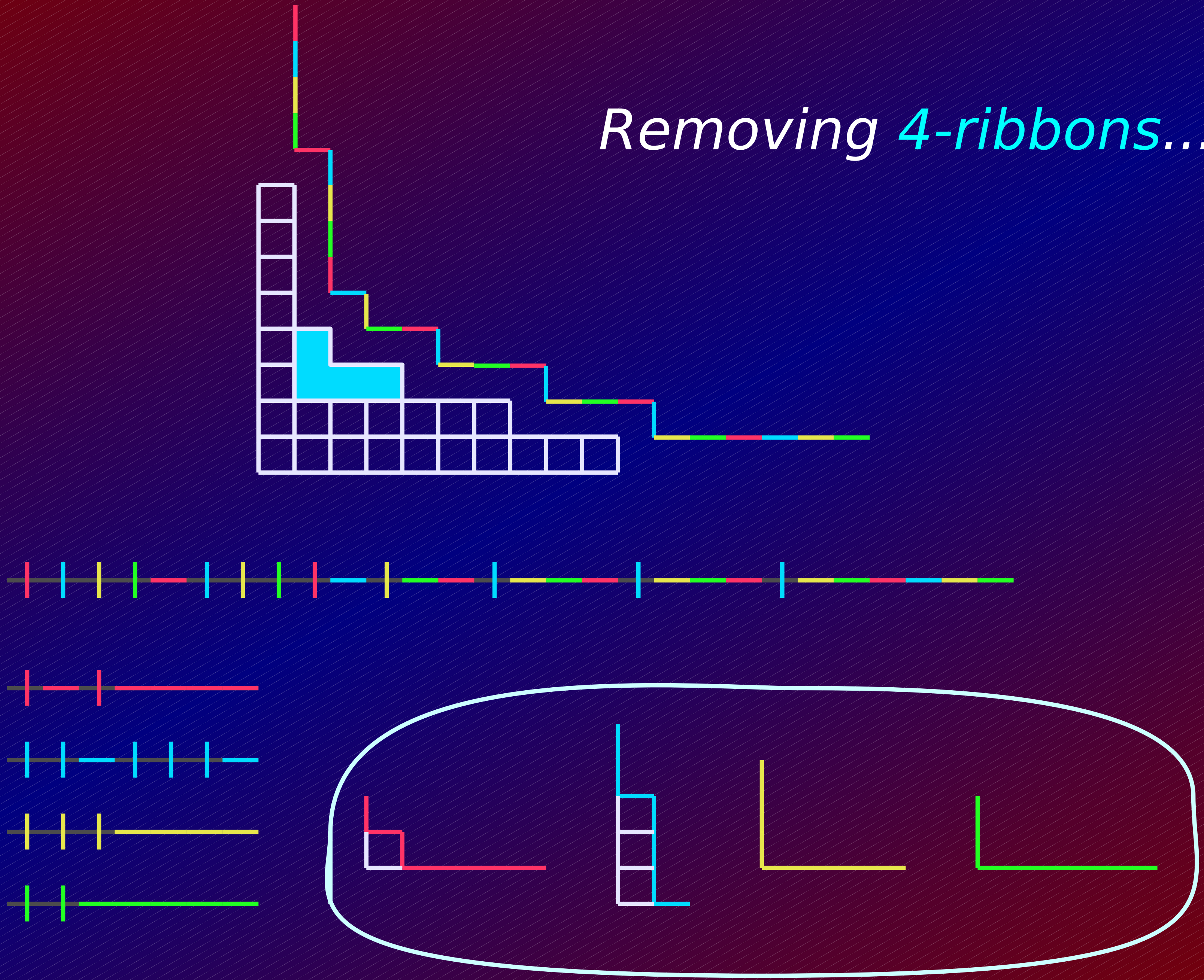

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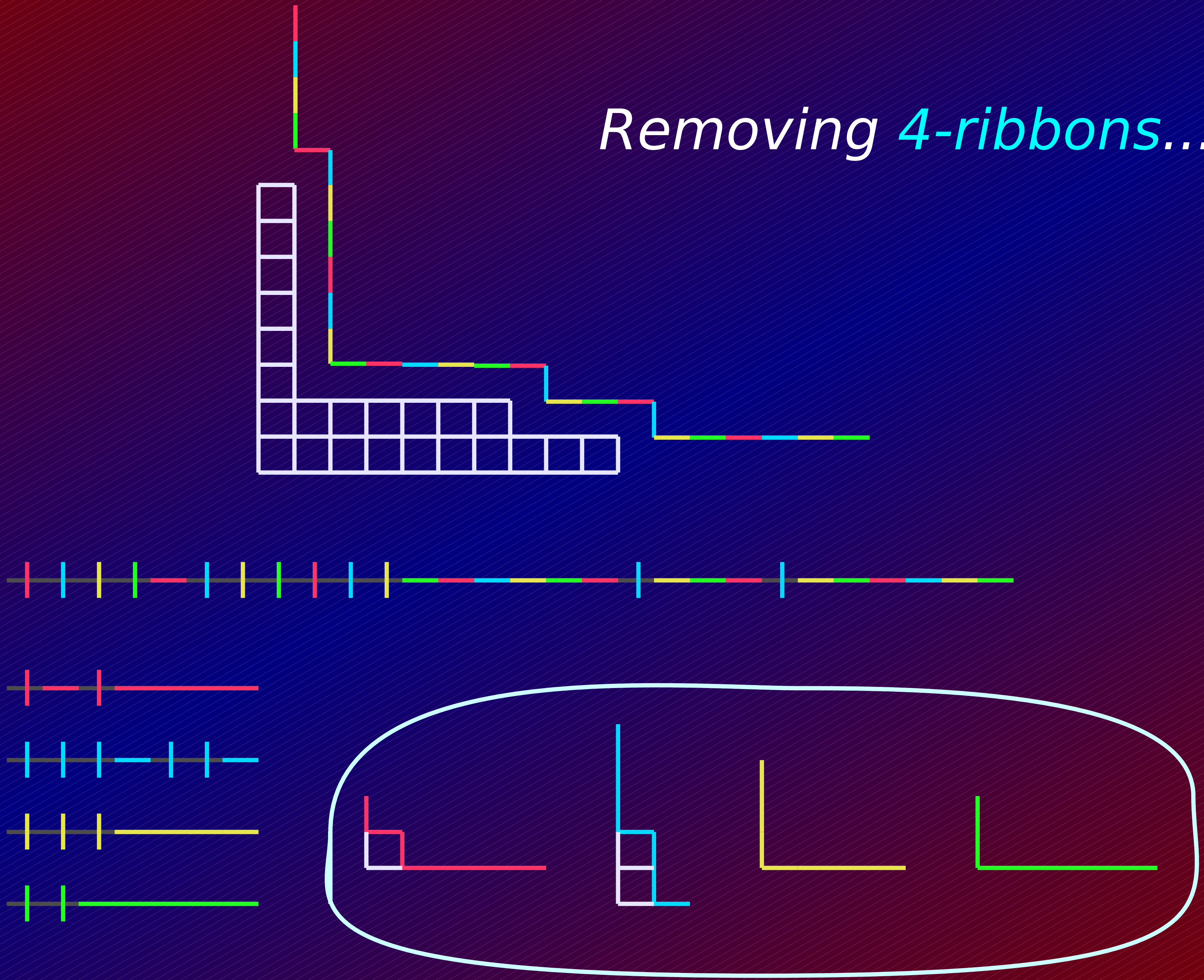

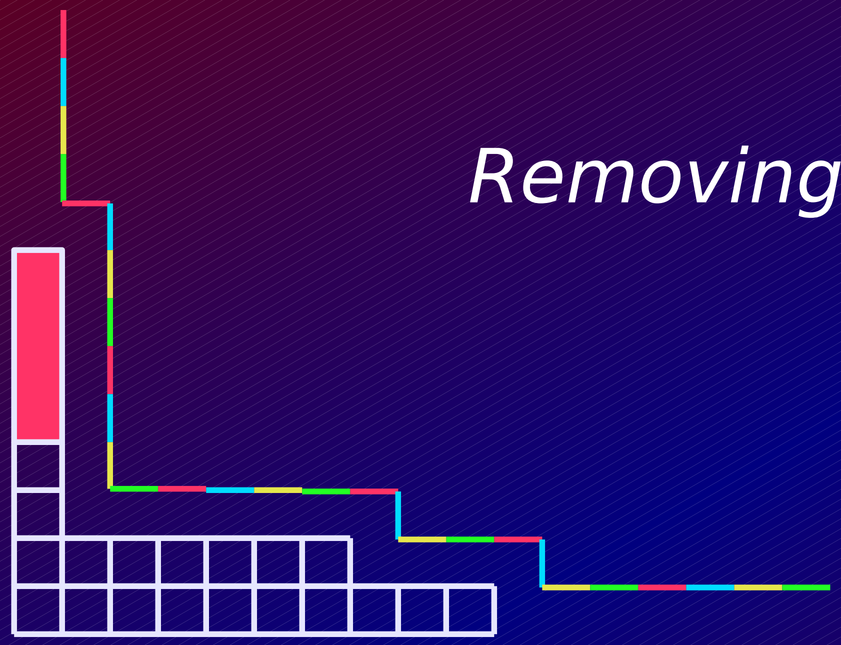

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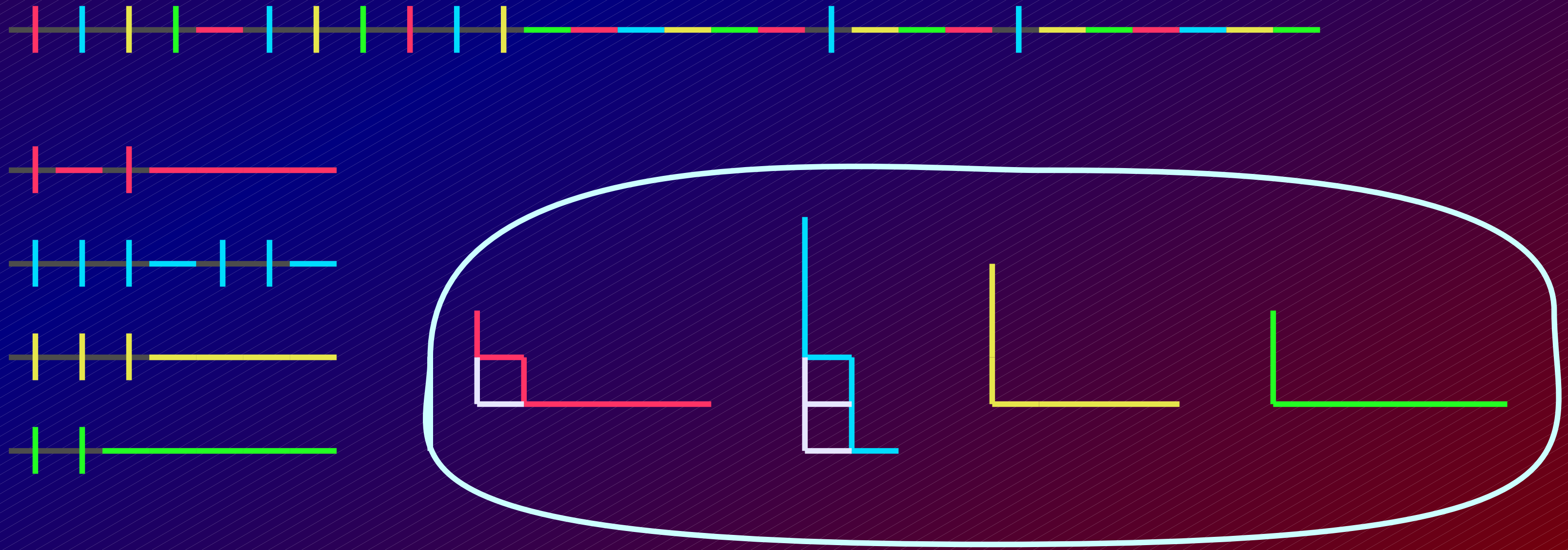

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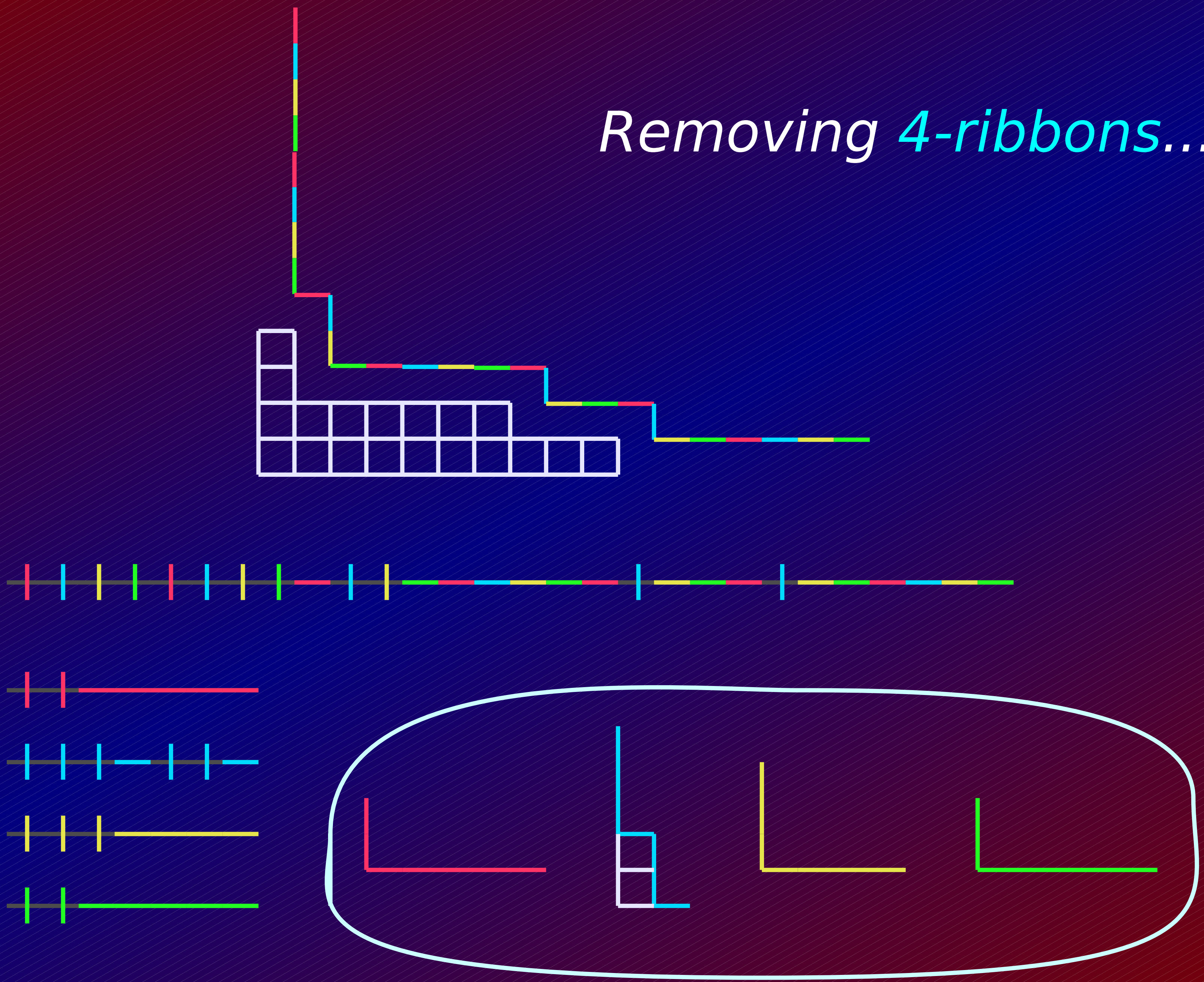

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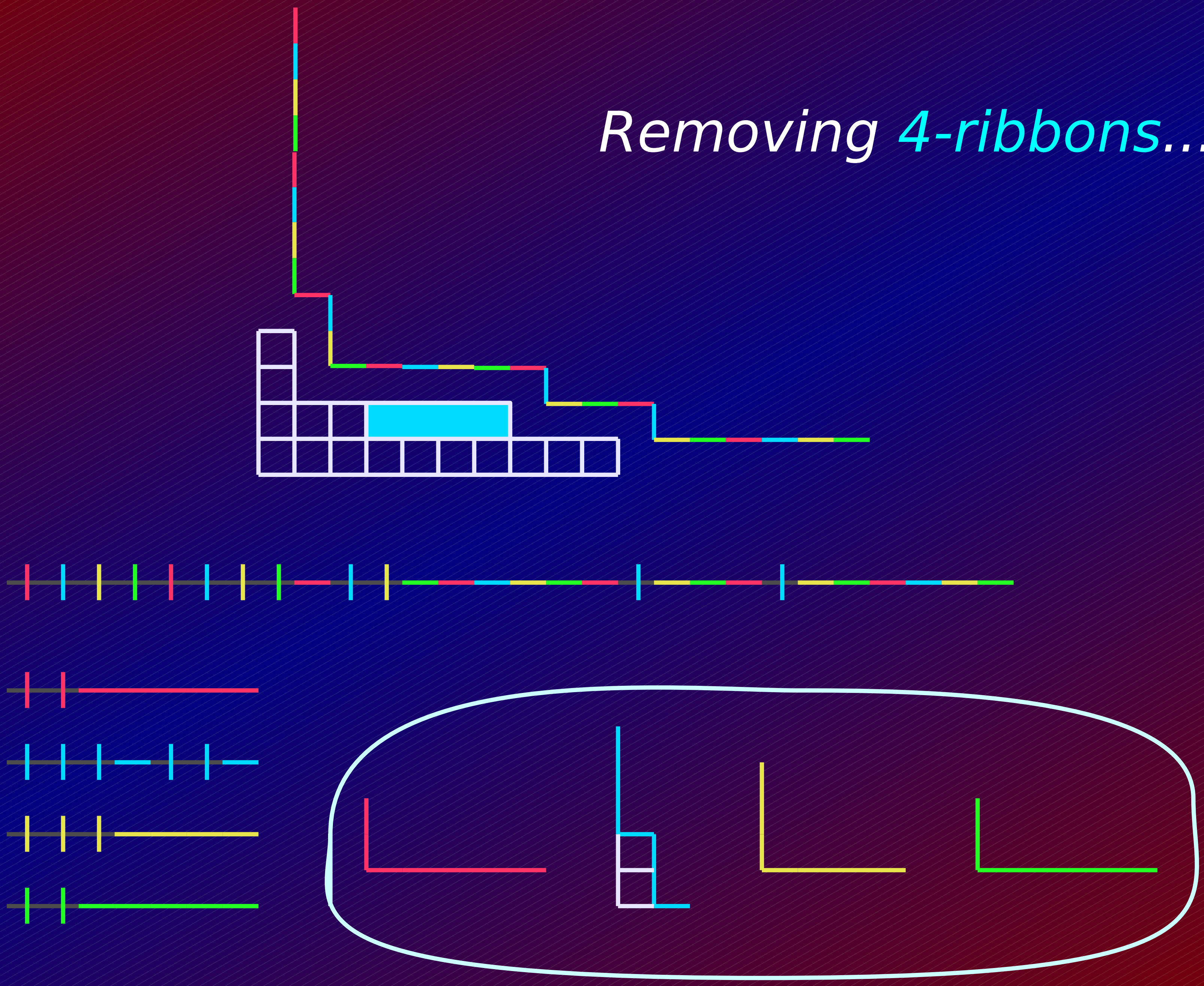

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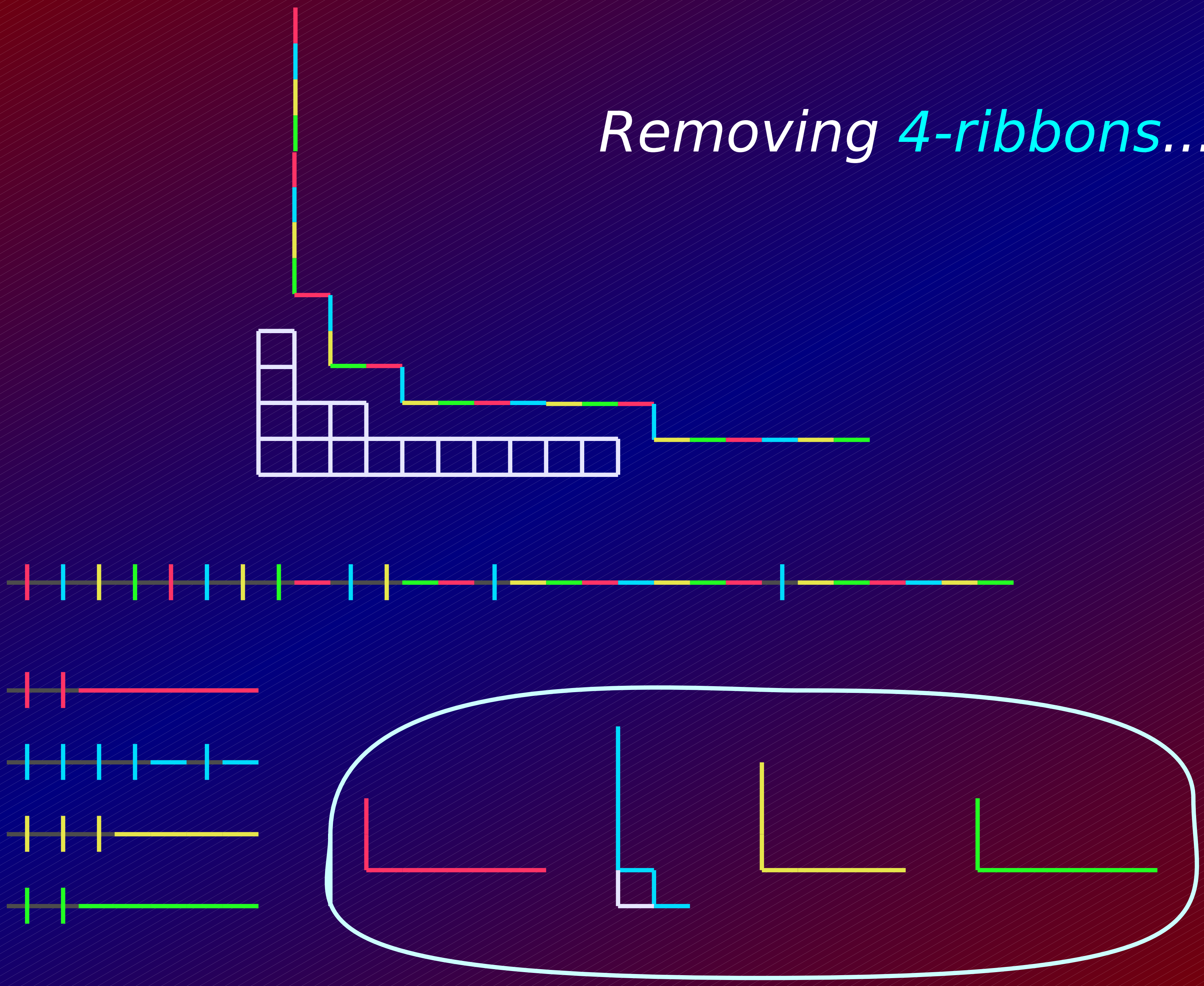

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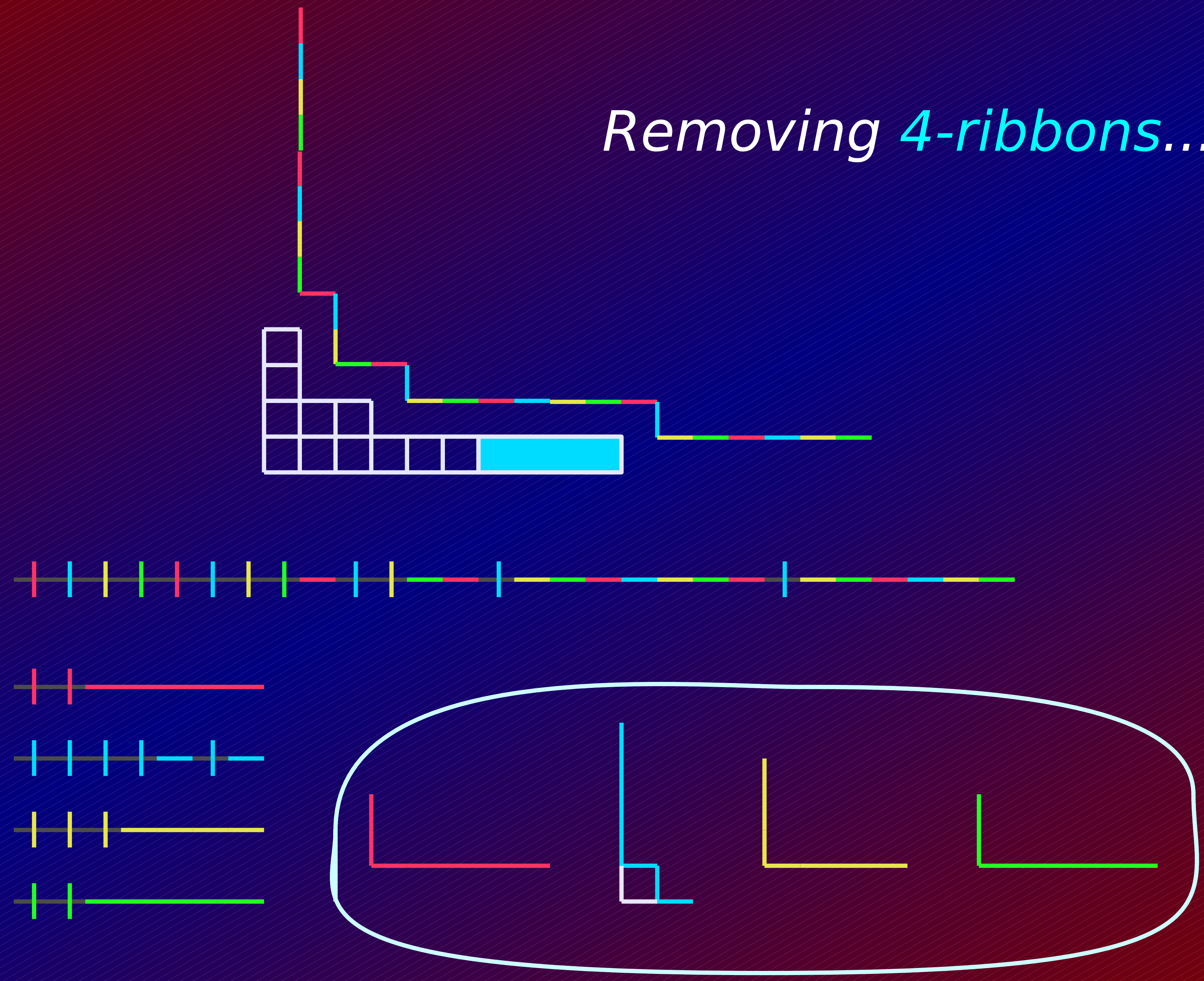
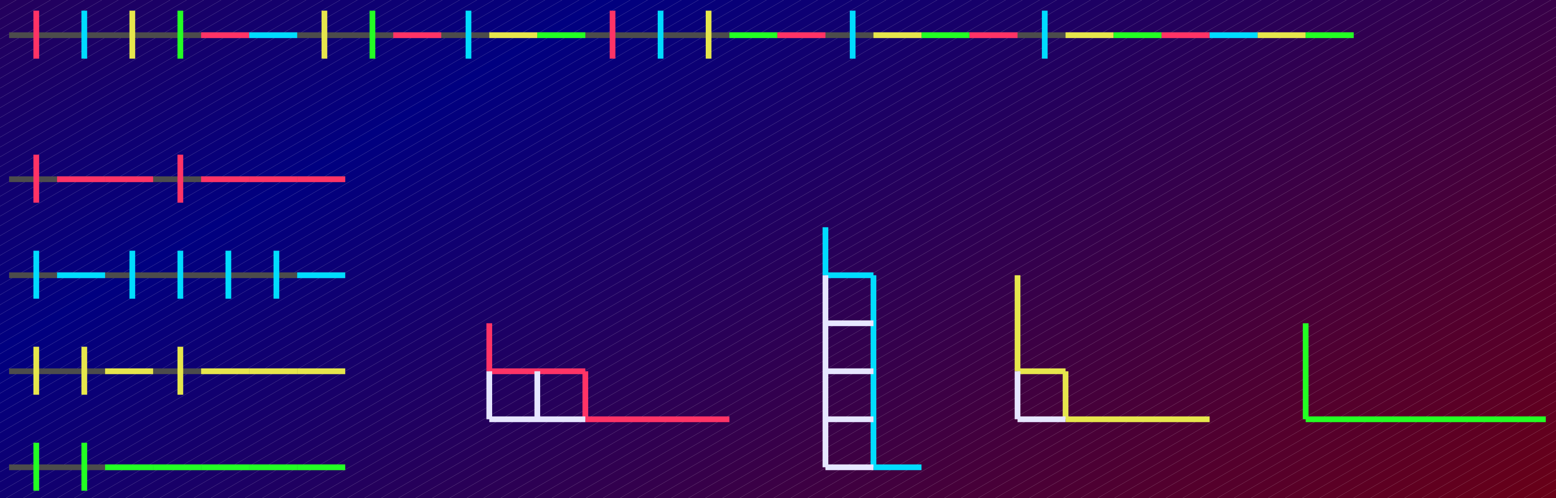


Removing *4-ribbons*...

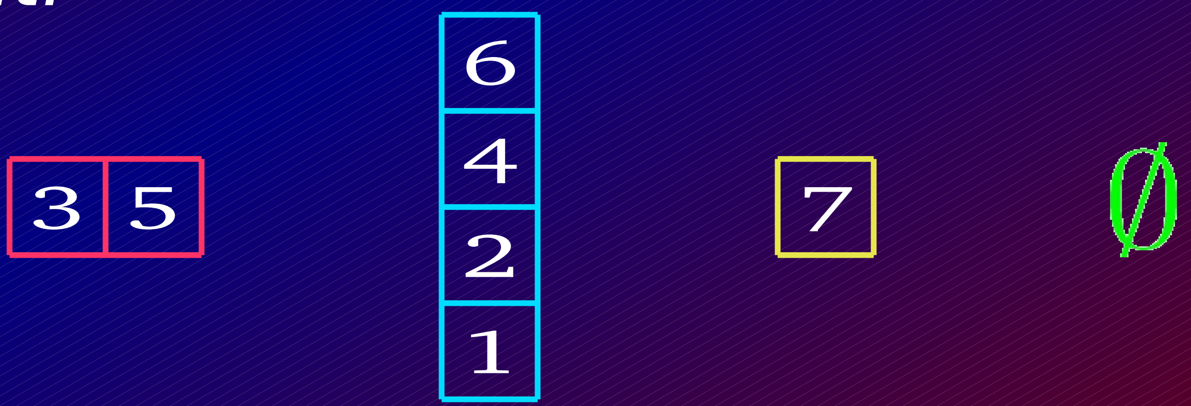

Removing 4-ribbons...
What's left is the 4-core

Remembering the order gives a ribbon tableau…



Remembering the order gives a ribbon tableau...

...and corresponding standard tableau on the k-quotient.



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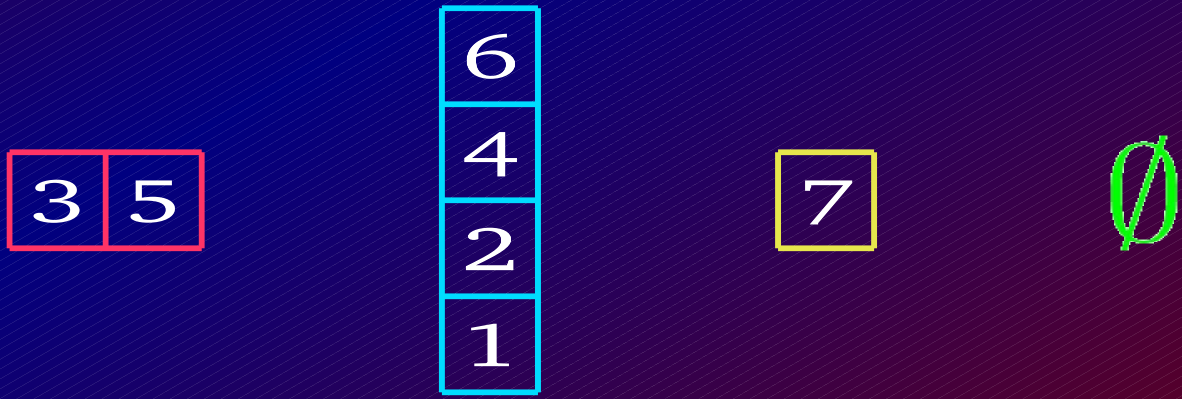
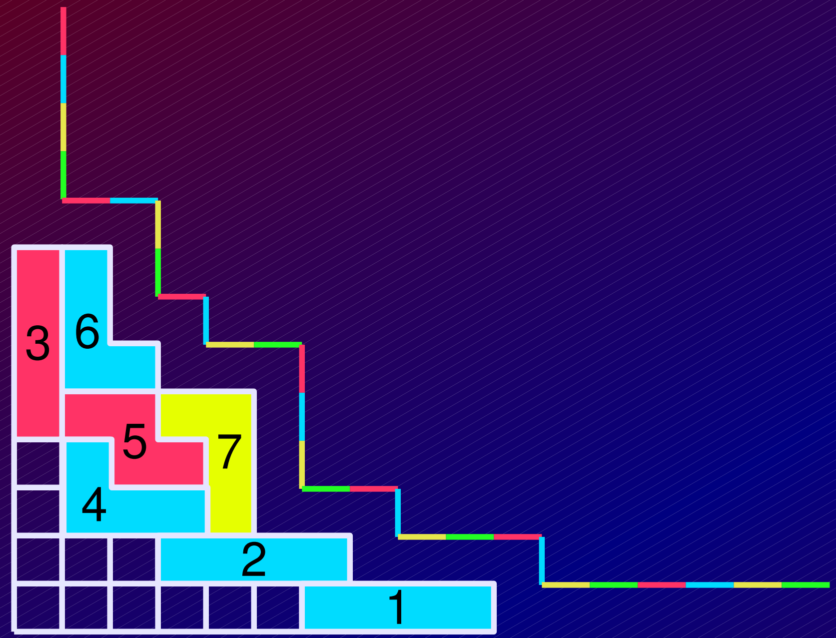


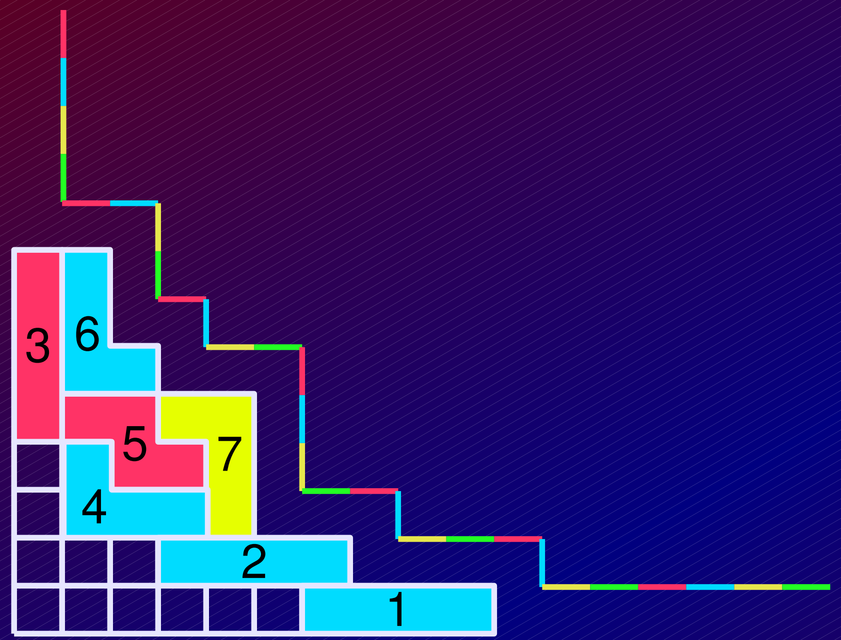
This also works for semistandard tableaux.



The ***spin*** *of a ribbon tableau is defined to be*

$$sp(T) = \sum_R (h(R) - 1)$$

where $h(R)$ *denotes the height of a ribbon.*



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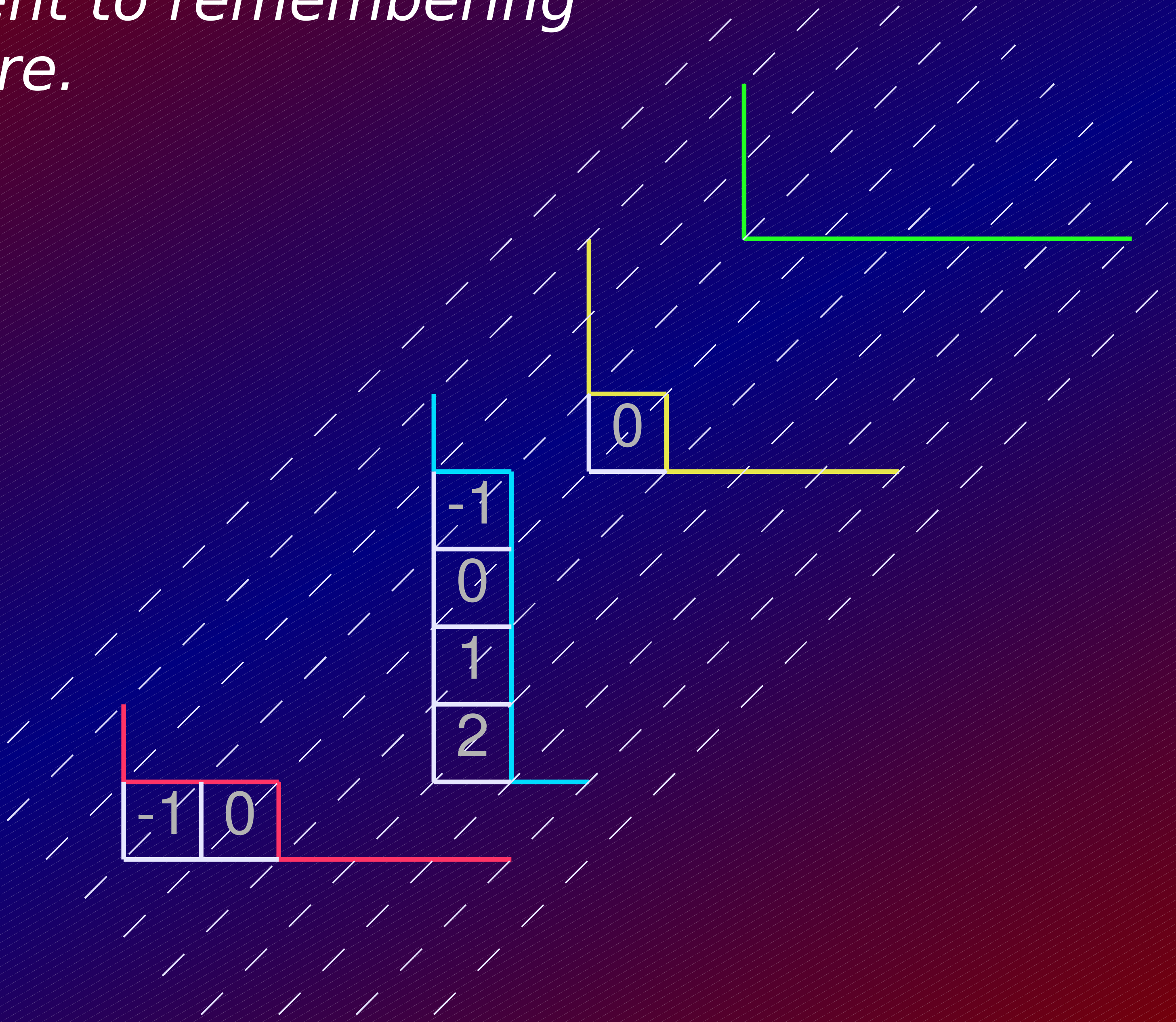
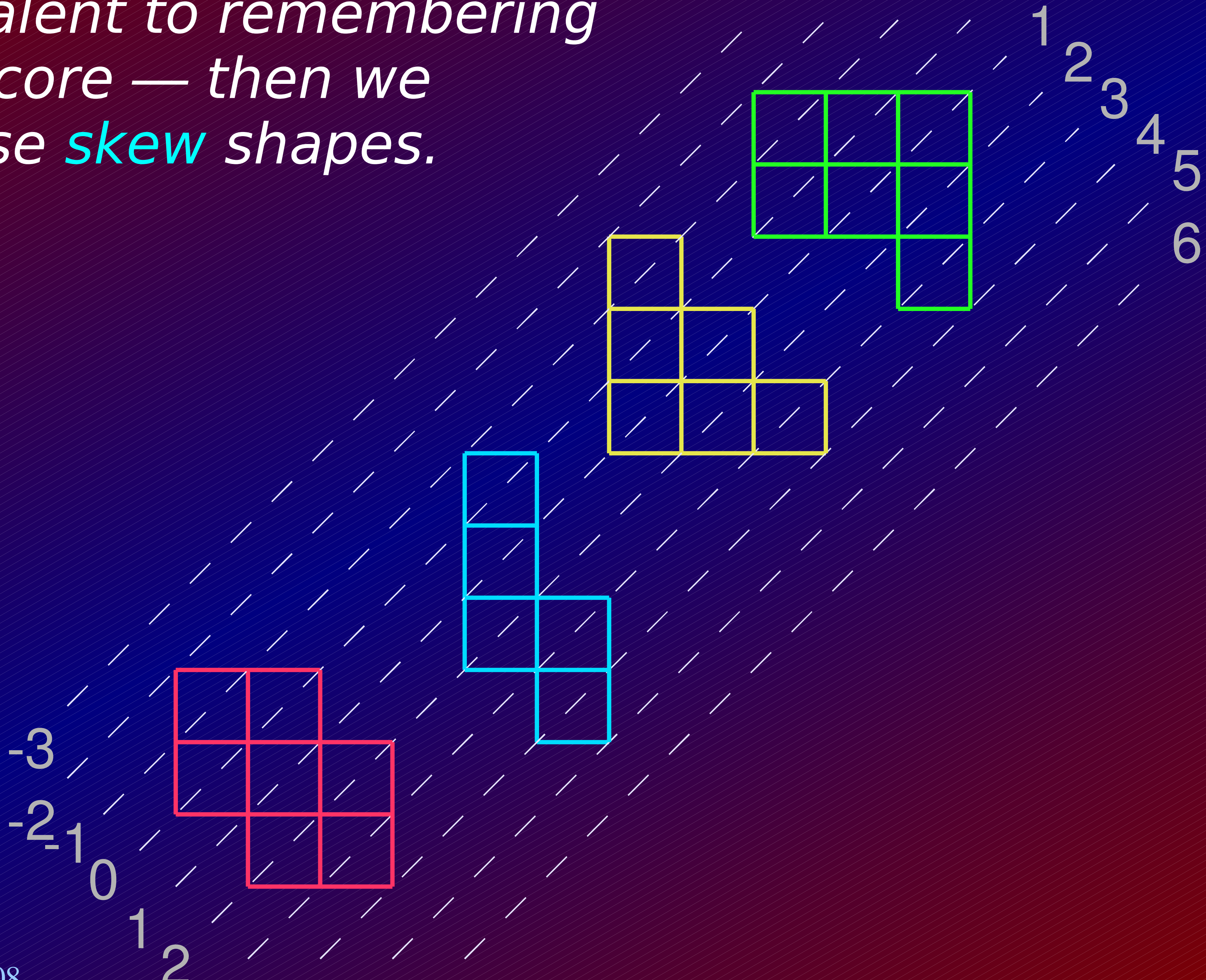
where $h(R)$ *denotes the height of a ribbon.*
In this example,

$$sp(T) = 0 + 0 + 1 + 1 + 2 + 2 + 3 = 9$$

Aligning content lines in the k-quotient is equivalent to remembering the k-core.



Aligning content lines in the k-quotient is equivalent to remembering the k-core — then we can use skew shapes.

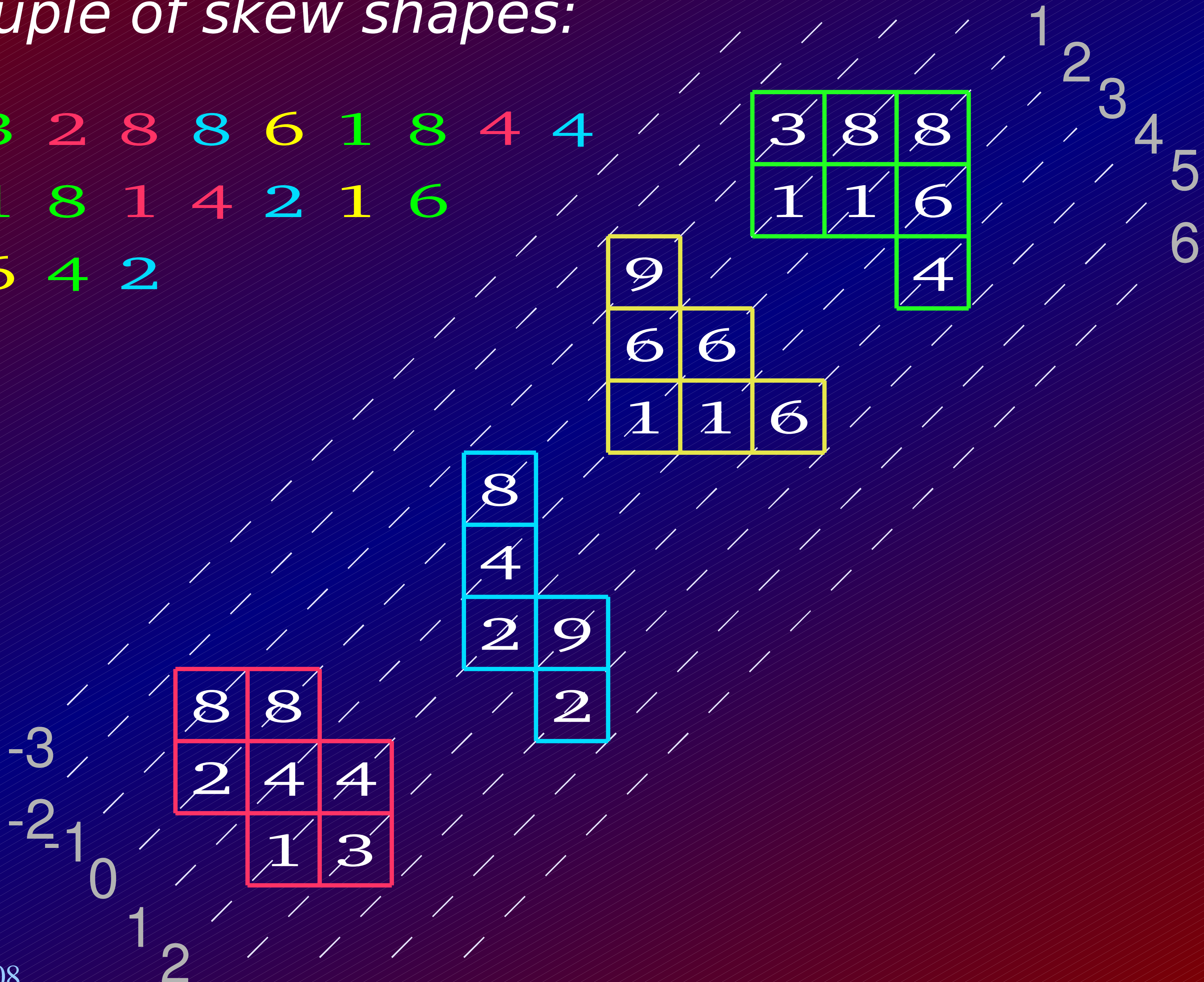


Content reading word of a tableau on a tuple of skew shapes:

8 9 3 2 8 8 6 1 8 4 4

1 6 1 8 1 4 2 1 6

3 9 6 4 2



We count inversions in the reading word if they are on the same line—

8 9 3 2 8 8 6 1 8 4 4

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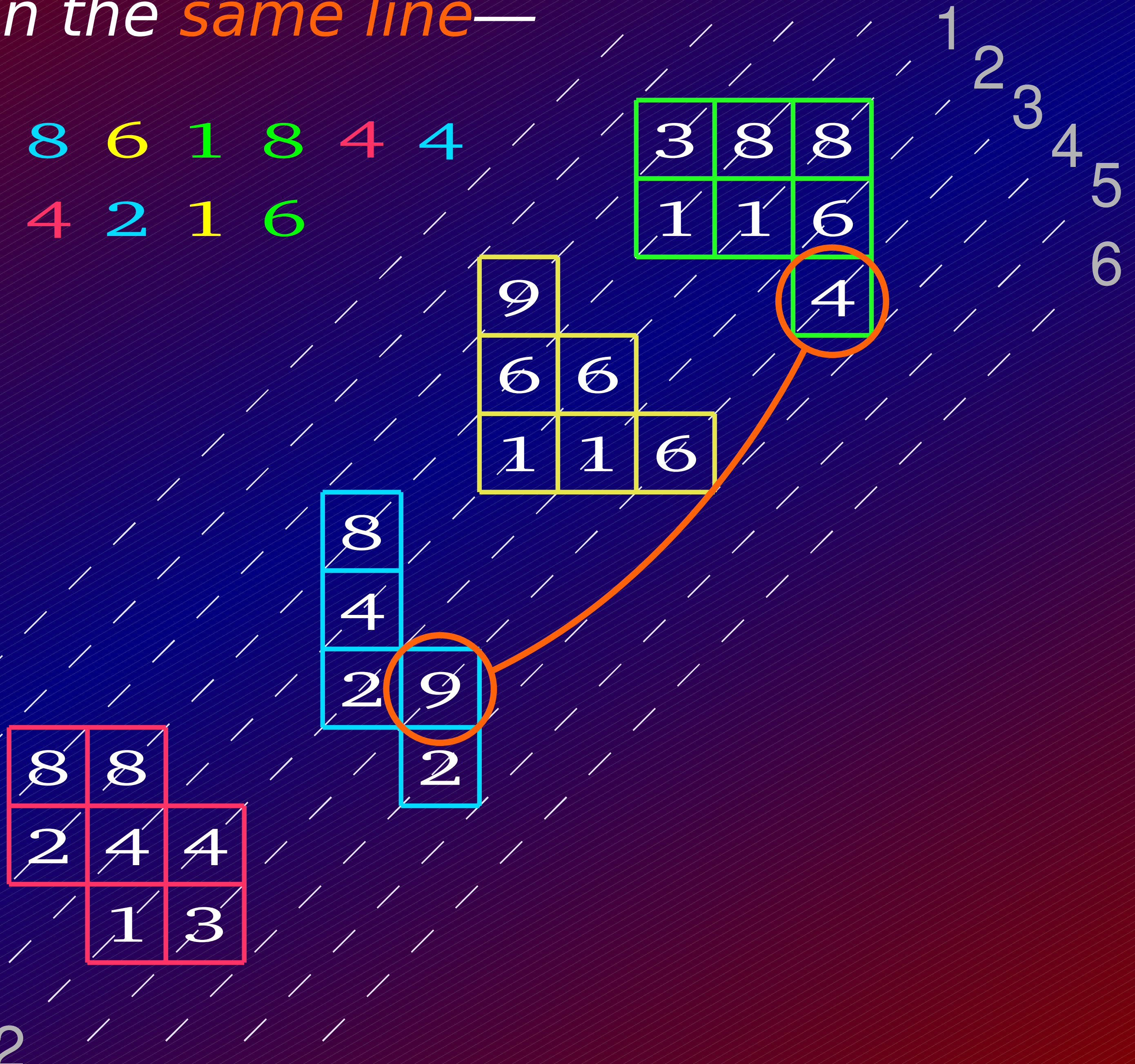

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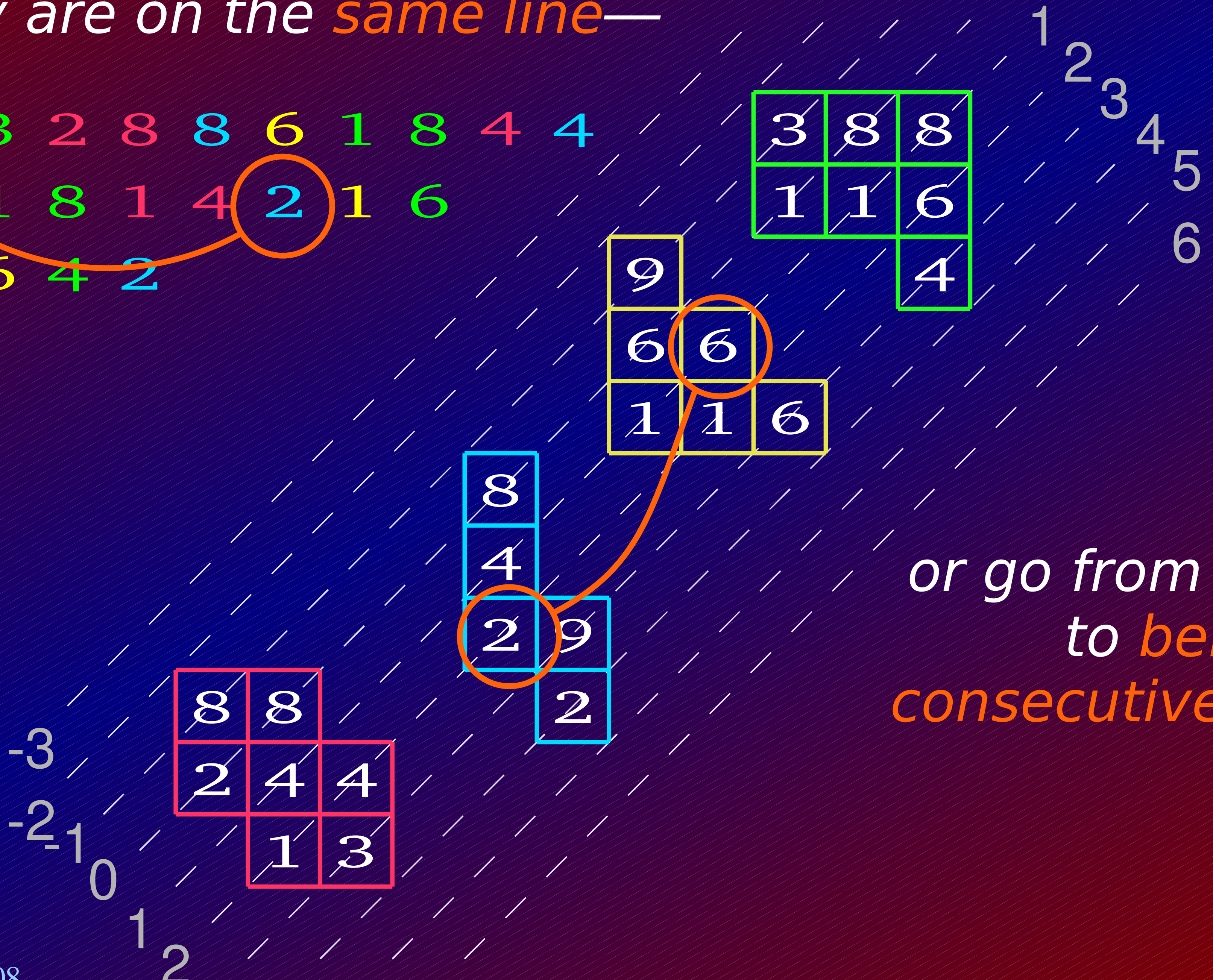
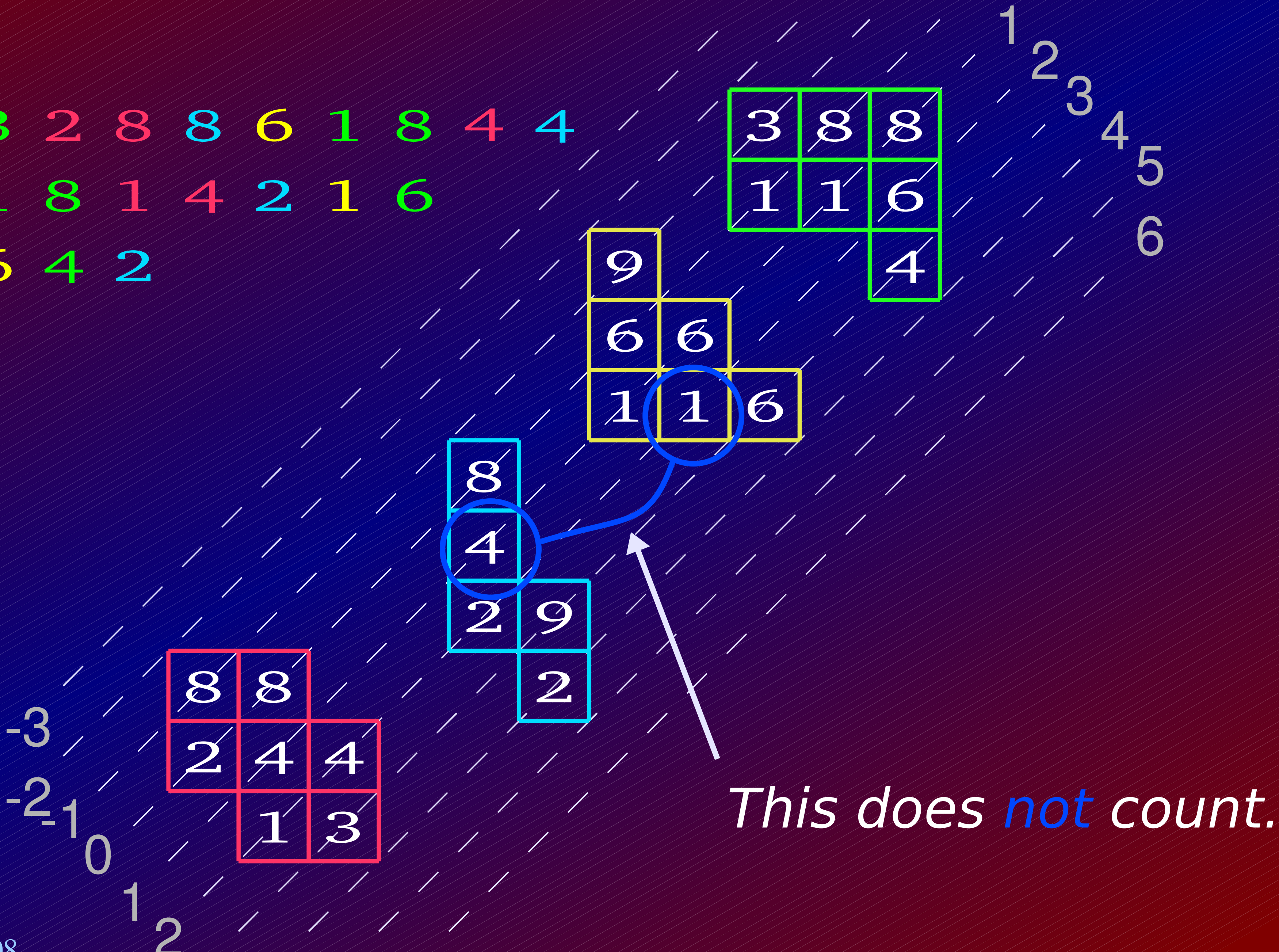


or go from above to below on consecutive lines.

8 9 3 2 8 8 6 1 8 4 4

1 6 1 8 1 4 2 1 6

3 9 6 4 2



Proposition. Let T be a k-ribbon tableau, and S the corresponding k-tuple of ordinary tableaux. Then

$$sp(T) = C - 2\,inv(S)$$

for a constant C depending only on the shape.

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Definition. *The* ***LLT polynomial*** *indexed by a tuple of (skew) shapes ν is the generating function*

$$G_\nu(x;\, q) = \sum_{SSYT(\nu)} q^{inv(S)}\, x^S$$

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Definition. *The **LLT polynomial** indexed by a tuple of (skew) shapes ν is the generating function*

$$G_\nu(x;\, q) = \sum_{SSYT(\nu)} q^{inv(S)}\, x^S$$

$$= \sum_{SSRT(\mu)} q^{-\frac{1}{2}(sp(T)-C)}\, x^T$$

3/20/08 **Mark Haiman** - "LLT polynomials"

(introduced by Francesco Brenti)

Bio: PhD at MIT with Gian Carlo Rota.
Then UCSD with Garsia.
Now at UC Berkeley

1^{st} half of talk on Power point
2^{nd} half on blackboard.

1^{st} half (Power point)

- defines k-cores & k-quotients via colorful example.
- spin of ribbon tableau $sp(T) = \sum_R (h(R) - 1)$
- content reading word, inversions
- $sp(T) = C - 2\,inv(S)$
- LLT poly indexed by a tuple of skew shapes ν is gen. fcn.
$$G_\nu(x;q) = \sum_{SSYT(\nu)} q^{inv(S)} x^S = \sum_{SSRT(\nu)} \cdots$$

2^{nd} half (Black board)

LLT's in Macdonald Theory

- Haglund: $\tilde{H}_\mu(x;q,t) = \sum_\nu q^{a(\nu)} t^{b(\nu)} G_\nu(x;q)$
ν: $\nu^{(i)}$ is a ribbon of size μ_i
- $R_n = \mathbb{C}[X_1, Y_1, \ldots, X_n, Y_n] / (S_n\text{-invariants})$

Frobenius Characteristic

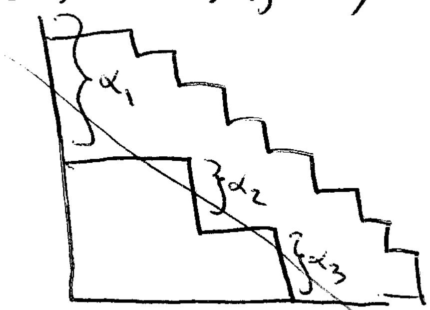
$$\mathcal{F}_{R_n}(x;q,t) \underset{\text{def}^n}{=} \sum_{i,j} q^i t^j \, F\,ch(R_n)_{i,j} \underset{\text{thm}}{=} \nabla e_n(x)$$

$$\nabla \widetilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \widetilde{H}_\mu$$

<u>Conjecture</u>: $\nabla e_n = \sum_{\lambda \subset (n-1,\dots,2,1)} t^{\binom{n}{2}-|\lambda|} G_\nu(x;q)$

(H., Haglund, Loehr, Remmel, Ulyanov) ↑ staircase shape

<u>eg</u>.



$\nu^{(i)} = (1^{\alpha_i})$

• $R(n,l)$ polygraph ring $= \mathbb{C}[\underline{X},\underline{Y},\underline{A},\underline{B}]/I$

$$\mathcal{F}_{R(n,l)}(x;q,t) = \Delta^l h_n\left[\frac{X}{(1-t)(1-q)}\right]$$

$$\Delta \widetilde{H}_\mu = B_\mu(q,t)\,\widetilde{H}_n$$

t	qt	
1	q	q^2

$\sum \to B_\mu$

<u>Conjecture</u> (H.) $= \sum_\nu t^{-} G_\nu\left(\frac{X}{1-q};q\right)$

$\nu^{(i)}$ columns

• <u>Conjecture</u> (Loehr, Warington)

$$\nabla s_\lambda(x) = (-1)^{d(\lambda)} \sum_\nu t^{-} G_\nu(x;q)$$

In Haglund's formula, if $\mu_1 \leq k$, then ν has "band width" $\leq k$.

<u>Conjecuture</u>: For any ν with b.w. $\leq k$, then $G_\nu(x;q) \in \mathbb{N}[q]\cdot\{A_\lambda^{(k)}(x;q)\}$ (this would imply $\widetilde{H}_\mu$ is too!)

$\mathcal{G}^{G,L}_{\alpha,\beta}(x;q) \in \left(\mathbb{N}[q^{\pm 1}] \cdot (G\text{-characters})\right)^{\wedge}$ ← to denote possibly infinite

$\alpha, \beta \in X_{++}(L)$ $\quad \alpha - \rho_L, \beta - \rho_L \longleftrightarrow$ tuples of partitions

$\nu \longleftrightarrow \beta/\alpha$

<u>What are some of the players?</u> $A = \mathbb{Q}[u^{\pm 1}]$ ground ring.

G - reductive alg. grp $\quad W_0$ Weyl grp

L - Levi subgrp $\hookrightarrow W_J$

Extended Affine Weyl Grp $W = W_0 \ltimes X \qquad X = \text{wts}(G)$

Hecke Algebra $\mathcal{H}$ has gen's $T_i = T_{s_i}$ $\quad T_w \quad w \in W$

$T_v T_w = T_{vw}$ if $l(v) + l(w) = l(vw)$

$(T_i - u)(T_i + u^{-1}) = 0 \qquad q = u^2$

$e^+ = \sum_{W_0} u^{l(w)} T_w$

$e_J^- = \sum_{W_J} (-u^{-1})^{l(w)} T_w$

We're interested in the module $e_J^- \mathcal{H} e^+$ for $Z(\mathcal{H}) = (A[Y^{\mu}])^{W_0}$ ← $\psi \; x_\lambda$

$\lambda \in X, \quad \tau(\lambda) \in W. \qquad \lambda \in X_+ \to Y^\lambda = T_{\tau(\lambda)} \xrightarrow{\text{extended}} Y^\mu, \; \mu \in X$

Look at $e_J^- T_w e^+ = 0$ if orbit $W_J w W_0$ not regular.

$\{e_J T_w e^+\}$ basis : $w \in (W_J \backslash W / W_0)_{reg}$

How to work with these double cosets $W_J w W_0$?

$(W_J \backslash W / W_0)_{reg} \longleftrightarrow (W_J \backslash X)_{reg} \longleftrightarrow X_{++}(L)$

another way $\longleftrightarrow$ See next page! $W_0 \backslash W / W_J$

Fix $k>0$. **Let** $W \curvearrowright X$ at "level $-k$" $\tau(\lambda)\cdot\beta = \beta - k\lambda$.

Fix $\eta \in -X_+$ s.t. (i) $\eta \in -k A_0$
(ii) $\mathrm{Stab}^W(\eta) = W_J$

Then $W/W_J \longleftrightarrow$ the orbit of η, $W\cdot\eta$

$$\Rightarrow (W_0\backslash W/W_J)_{reg} \longleftrightarrow W_0 \text{ orbits in } W\cdot\eta \longleftrightarrow X_{++}(G)\cap W\cdot\eta$$

Given $\mu \in W\cdot\eta$ $\quad \mu = -k\lambda + v(\eta)$ $\quad v \in W_0$ min in vW_S

$$V_\mu = \overline{Y^\lambda}\, T_v\, e_J^- \in \mathcal{H} e_J^- .$$

"Physics notation" (inherited from Leclerc & Thibon) $\rightarrow$ $|\mu\rangle = e^+ V_\mu \in e^+ \mathcal{H} e_J^-$

$\uparrow$ $\mu \in X_{++}(G)$ - basis

$\langle \alpha_i^\vee, \mu\rangle = -m < 0 \qquad i \neq 0$

$$|\mu\rangle = \begin{cases} -|s_i\mu\rangle & \text{if } k\mid m \\ u^{-1}|s_i\mu\rangle + u^{-1}|u + r\alpha_i\rangle - |s_i\mu - r\alpha_i\rangle & \text{if } r \equiv m \pmod k \\ u^{-1}|s_i\mu\rangle & \text{otherwise} \end{cases}$$

Consequence of this:

matrix coeff $\langle\lambda|\, x_\gamma\, |\mu\rangle = q^{?}\langle S_\gamma\rangle\, G_{\beta/\alpha}(x;q)$ $\quad$ (? depends on λ & μ)

$$W\cdot\eta \cap X_{++}(G) \ni \mu \longleftrightarrow \beta \in X_{++}(L)$$
$$\ldots \longleftrightarrow \alpha$$

$$\langle e_J^- T_{v^{-1}} e^+\rangle\, \chi_\gamma(Y)\, \langle e_J^- T_{w^{-1}} e^+\rangle$$

v min rep of $W_J \tau(\alpha) W_0$

w " " " " $\tau(\beta)$ "

Out of Time ...

He speaks of how Littlewood-Richardson coeff's appear in the latter description.

Audience Questions

End of Lecture