Q systems as cluster algebras

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1. A combinatorial identity

2. A sequence of inequalities

3. Q-systems as cluster algebras and the polynomial property

A combinatorial identity

- Fix a set of integers $\mathbf{n} = (n_1, ..., n_k) \in \mathbb{Z}_+^k$ and $\ell \in \mathbb{Z}_+$.

Theorem

$$N_{\mathbf{n},\ell}^{(k)} := \sum_{\mathbf{m}\in\mathbb{Z}_+^k} \prod_{i=1}^{k} \binom{m_i + p_i(\mathbf{m})}{m_i} = \sum_{\substack{\mathbf{m}\in\mathbb{Z}_+^k \\ p_i(\mathbf{m})\geq 0}} \prod_{i=1}^{k} \binom{m_i + p_i(\mathbf{m})}{m_i} =: M_{\mathbf{n},\ell}^{(k)}$$

- $p_i(\mathbf{m}) = \sum_{j=1}^{k} \min(i,j)(n_j - 2m_j)$,
- The sums are both taken so that $\ell = \sum_{i=1}^{k}(in_i - 2im_i)$ is fixed.
- The binomial coefficient is defined for $p \in \mathbb{Z}$

$$\binom{m+p}{m} = \frac{(p+m)(p+m-1)\cdots(p+1)}{m!}.$$

- For any simple Lie algebra $\mathfrak{g}$ of rank r, any $\lambda \in P^+$ and $\mathbf{n} \in \mathbb{Z}_+^{r\times k}$, there is a similar identity:

$$M_{\mathbf{n},\lambda}^{(k)} = N_{\mathbf{n},\lambda}^{(k)}$$

Where does this identity come from?

- Left hand side: $N_{\mathbf{n},\ell} = \lim_{k\to\infty} N^{(k)}_{\mathbf{n},\ell}$ is known to be the dimension of the space
$$N_{\mathbf{n},\ell} = \left| \mathrm{Hom}_{\mathfrak{sl}_2}(\bigotimes_i V(i\omega_1)^{\otimes n_i}, V(\ell\omega_1)) \right|.$$
$V(i\omega_1)$ is the $i+1$-dimensional $\mathfrak{sl}_2$-module
- The right hand side $M_{\mathbf{n},\ell} = \lim_{k\to\infty} M^{(k)}_{\mathbf{n},\ell}$ is the number of "string solutions" to the Bethe equations in the $\mathfrak{sl}_2$-inhomogeneous Heisenberg spin chain (an exactly solvable model in statistical mechanics).
- The identity
$$M_{\mathbf{n},\ell} = N_{\mathbf{n},\ell}$$
is the **completeness hypothesis**: Bethe vectors span the Hilbert space of the Hamiltonian.
- There is such a model for any simple Lie algebra, for a certain class of $\mathfrak{g}$-modules. In general, completeness is the **combinatorial Kirillov-Reshetikhin conjecture.**

How to prove the $M = N$ identity

- An identity of the form

$$\sum_{\mathbf{m}:\text{Restrictions}} f(\mathbf{m}) = \sum_{\mathbf{m}} f(\mathbf{m})$$

  is proven by relaxing the restrictions and looking at properties of generating functions.

- Define a generating function

$$Z^{(k)}_{\mathbf{n},\ell}(t;\mathbf{u}) = \sum_{\mathbf{m}} t^{-q} \prod_{i=1}^{k} \binom{m_i + p_i + q}{m_i} u_i^{p_i+q}, \quad q = \ell - \sum_i i(n_i - 2m_i)$$

- $N^{(k)}_{\mathbf{n},\ell}$ = The constant term of $Z_{\mathbf{n},\ell}(t; 1, ..., 1)^{(k)}$ in t.
- $M^{(k)}_{\mathbf{n},\ell}$ = The constant term in t of **only the positive powers in** $\{u_i\}$ of $Z_{\mathbf{n},\ell}(t;\mathbf{u})^{(k)}$, evaluated at $u_1 = ... = u_k = 1$.

A theorem about generating functions

Theorem (Di Francesco, K.)

1. *The generating function factorizes:*

$$Z^{(k)}_{\mathbf{n},\ell}(t;\mathbf{u}) = \frac{\mathcal{Q}_1 \mathcal{Q}_k^{\ell+1}}{\mathcal{Q}_{k+1}^{\ell+1}} \prod_{j=1}^{k} u_i^{-1} \mathcal{Q}_j^{n_j}$$

   where the functions $\mathcal{Q}_m$ are defined by the recursion:

$$\mathcal{Q}_{m+1}(t;\mathbf{u})\mathcal{Q}_{m-1}(t;\mathbf{u}) = \frac{\mathcal{Q}_m(t;\mathbf{u})^2 - 1}{u_m}, \quad \mathcal{Q}_0 = 1,\ \mathcal{Q}_1 = t.$$

2. $M^{(k)}_{\mathbf{n},\ell} = N^{(k)}_{\mathbf{n},\ell}$ *if the first $k+1$ solutions to this system are* **polynomials in** t *when $u_1 = ... = u_k = 1$.*

Polynomial solutions

- The recursion relation when $u_i = 1$ is well-known:

$$Q_{m+1}Q_{m-1} = Q_m^2 - 1, \quad Q_0 = 1, \quad Q_1 = t$$

  The solutions are **Chebyshev polynomials** of the second kind in t, so $M = N$ in this case.
- This recursion is also satisfied by the characters of the irreducible representations of $\mathfrak{sl}_2$, where Q_1 is the character of the fundamental representation and Q_0 of the trivial rep. Polynomiality follows from this fact.
- This is the simplest "Q-system" which appeared more generally in the work of Kirillov-Reshetikhin in the 80's.

Theorem

The Q-system with no boundary conditions on Q_0, Q_1 is a cluster algebra with seed

$$\mathbf{x} = (iQ_0, iQ_1), \quad B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

so the solutions are Laurent polynomials in Q_0, Q_1 **except** *at the "KR point" $Q_0 = 1$, where they are polynomials in Q_1.*

The general picture: Cartan matrix of finite type

Feigin-Loktev
fusion product
↓

$$|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda)\right)|$$

Chari's
$\mathfrak{g}[t]$-modules
↓

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| \leq |\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda)\right)| \geq M_{\mathbf{n},\lambda}$$

Restricted
↙ Sum

$$\vee|$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

↑
$U_q(\widehat{\mathfrak{g}})$-modules

↖
Unrestricted
Sum

The general picture: Cartan matrix of finite type

Feigin-Loktev fusion product ↓

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda)\right)\right|$$

Chari's $\mathfrak{g}[t]$-modules ↓

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| \leq \left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda)\right)\right| \leq M_{\mathbf{n},\lambda}$$

Restricted Sum ↙ $M_{\mathbf{n},\lambda}$

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| \geq \left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

↑ $U_q(\widehat{\mathfrak{g}})$-modules

↖ Unrestricted Sum

- $KR_{\alpha,m}(\zeta)$ $1 \le \alpha \le r = \mathrm{rank}(\mathfrak{g})$, $m \in \mathbb{Z}_+$, $\zeta \in \mathbb{C}^*$ is a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module.
- Introduced by Kirillov and Reshetikhin in the definition of the generalized Heisenberg spin chain model. (See Kuniba, Nakanishi, Suzuki 93 for formal definition)
- KR-modules are characterized by the Drinfeld polynomials $(P_1, ..., P_r)$

$$P_\beta = \begin{cases} 1, & \beta \neq \alpha \\ \prod_{i=0}^{m-1}(1 - q_\beta^{2i}\zeta u), & \beta = \alpha. \end{cases}$$

- Decomposition (of tensor products of KR-modules) into irreducible components under this restriction is given by tensor product multiplicities $N_{\mathbf{n},\lambda}$ ($n_{\alpha,m} = \#$ modules of type $KR_{\alpha,m}$) **Explicit combinatorial formula.** (Proved by Kirillov-Reshetikhin for A_n, more generally follows from theorems of HKOTY, Nakajima, Hernandez).

The general picture: Cartan matrix of finite type

Feigin-Loktev fusion product

$$|\mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}^*_{\mathbf{n}}\,,\,V(\lambda))|$$

Chari's $\mathfrak{g}[t]$-modules

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\,V(\lambda)\right)\right| \leq |\mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}^*_{\mathbf{n}}\,,\,V(\lambda))| \leq M_{\mathbf{n},\lambda}$$

Restricted Sum ($M_{\mathbf{n},\lambda}$)

$$\vee|$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\,V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

$U_q(\widehat{\mathfrak{g}})$-modules

Unrestricted Sum ($N_{\mathbf{n},\lambda}$)

The N-sum formula

For each $\mathbf{n} \in \mathbb{Z}_+^{r\times k}$ and $\lambda \in P^+$ a dominant weight of $\mathfrak{g}$ (simply-laced, rank r), $N_{\mathbf{n},\lambda} := \lim_{k\to\infty} N_{\mathbf{n},\lambda}^{(k)}$ is an **unrestricted** sum of **alternating sign terms:**

$$N_{\mathbf{n},\lambda}^{(k)} = \sum_{\mathbf{m}\in\mathbb{Z}_+^{r\times k}} \prod_{\alpha=1}^{r} \prod_{i=1}^{k} \binom{m_{\alpha,i} + p_{\alpha,i}(\mathbf{m})}{m_{\alpha,i}}.$$

The sum is taken over $\mathbf{m}$ such that $\sum_i i(n_{\alpha,i} - \sum_\beta C_{\alpha,\beta} m_{\beta,i}) = h_\alpha(\lambda)$ and

$$p_{\alpha,i}(\mathbf{m}) = \sum_j \min(i,j) n_{\alpha,j} - \sum_{\alpha,j} C_{\alpha,\beta} \min(i,j) m_{\beta,j}.$$

Theorem

(Kirillov-Reshetikhin, HKOTY, Nakajima, Hernandez) $N_{\mathbf{n},\lambda}$ is equal the tensor product multiplicity for KR-modules for any simple Lie algebra.

The general picture: Cartan matrix of finite type

Feigin-Loktev fusion product

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda)\right)\right|$$

Chari's $\mathfrak{g}[t]$-modules

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| \leq \left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda)\right)\right| \leq M_{\mathbf{n},\lambda}$$

Restricted Sum

$$\vee|$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

$U_q(\widehat{\mathfrak{g}})$-modules

Unrestricted Sum

Chari's Kirillov-Reshetikhin modules $C_{\alpha,m}(\zeta)$

- Modules of the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$. "Classical limit" $q \to 1$ of KR-modules.
- Cyclic modules, defined by generators and relations: $C_{\alpha,m}(\zeta)$ is the cyclic module generated by $U(\mathfrak{g}[t])$ acting on the cyclic vector v subject to the relations

$$\begin{aligned} & x \otimes (t-\zeta)^n v = 0, \quad x \in \mathfrak{g}, n > 0 \\ & f_\beta v = 0, \quad \beta \neq \alpha (\text{simple roots}) \\ & f_\alpha^{m+1} v = 0 \\ & h_\beta v = \delta_{\alpha,\beta} m v. \end{aligned}$$

  $C_{\alpha,m}$ is the associated graded space wrt degree in t.
- The inequality $\mathrm{Hom}_{U_q(\mathfrak{g})}(KR_{\alpha,m} \, , \, V(\lambda)) \leq \mathrm{Hom}_{\mathfrak{g}}(C_{\alpha,m} \, , \, V(\lambda))$ follows from general deformation arguments.
- The equality was shown for the classical algebras by Chari.

The general picture: Cartan matrix of finite type

Feigin-Loktev fusion product

$$|\mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}^*_{\mathbf{n}}\,,\ V(\lambda))|$$

Chari's $\mathfrak{g}[t]$-modules

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\ V(\lambda)\right)\right|$$

Restricted Sum

$$M_{\mathbf{n},\lambda}$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\ V(\lambda)\right)\right| \quad = \quad N_{\mathbf{n},\lambda}$$

$U_q(\widehat{\mathfrak{g}})$-modules

Unrestricted Sum

Feigin-Loktev fusion products

- The fusion product $\mathcal{F}^*_{\{V_i\}}$ is Defined for a set of cyclic $\mathfrak{g}[t]$-modules $\{V_i(\zeta_i)\}$ with cyclic vectors v_i, localized at points $\zeta_i \in \mathbb{C}^*$.
- as $\mathfrak{g} \subset \mathfrak{g}[t]$-modules,

$$V_1 \otimes \cdots \otimes V_N \underset{g-\mathrm{mod}}{\simeq} U(\mathfrak{g}[t]) v_1 \otimes \cdots \otimes v_N$$

  if $\zeta_i \neq \zeta_j$ for all $i \neq j$.
- The left-hand side has a filtration by degree in t. The associated graded space is the fusion product $\mathcal{F}^*_{\{V_i\}}(\boldsymbol{\zeta})$.
- The graded components of $\mathcal{F}^*_{\{V_i\}}$ are $\mathfrak{g}$-modules, with graded multiplicities

$$\mathcal{M}_{\{V_i\},\lambda}(t) = \sum_{i \geq 0} \left| \mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}^*_{\{V_i\}}[i]\ ,\ V(\lambda)) \right| t^i.$$

Feigin Loktev conjecture about fusion coefficients

- The polynomials $\mathcal{M}_{\{V_i\},\lambda}(t)$ have non-negative coefficients by definition.
- **Theorems:** [K. 04] For tensor products of symmetric power representations of A_n, $\mathcal{M}_{\{V_i\},\lambda}(t)$ are the co-charge **Kostka polynomials**. For tensor products of rectangular highest-weight representations they are generalized Kostka polynomials of [Schilling, Warnaar]. [Ardonne, K., Stone 06].
- **Feigin-Loktev conjecture:** when the modules $V_i(\zeta_i)$ are "sufficiently well-behaved", the polynomials $\mathcal{M}_{\{V_i\},\lambda}(t)$ are independent of $\{\zeta_i\}$. (Proved in special cases.)
- In particular, this means that $\mathcal{M}_{\{V_i\},\lambda}(1)$ is equal to the tensor product multiplicity in the cases where the FL conjecture holds.
- In cases where we can compute fusion coefficients explicitly, and show they are independent of the spectral parameters, this provides a proof of the FL conjecture.

Fusion products of Chari's KR-modules

Theorem

(Ardonne, K., 2007)

1. *When all the modules* $V_i(\zeta_i)$ *are Chari's KR-modules, there is an inequality*

$$\mathcal{M}_{\{V_i\},\lambda}(t) \leq M_{\mathbf{n},\lambda}(t),$$

($n_{\alpha,m} = \#$ modules of type $C_{\alpha,m}$) where $M_{\mathbf{n},\lambda}(t)$ *is a polynomial expressed as a combinatorial fermionic (non-negative coefficients) formula (appeared in KR, HKOTY, Kirillov, Schilling, Shimozono, ...).*

2. *The equality holds iff* $M_{\mathbf{n},\lambda} = M_{\mathbf{n},\lambda}(1)$ *is equal to*

$$\left| \mathrm{Hom}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}} \, , \; V(\lambda) \right) \right| .$$

3. *The inequality* $\left| \mathrm{Hom}(\bigotimes_i V_i(\zeta_i) \, , \; V(\lambda)) \right| \leq \left| \mathrm{Hom}(\mathcal{F}^*_{\{V_i\}} \, , \; V(\lambda)) \right|$ *follows from a general deformation argument. [Feigin, Jimbo, K., Loktev, Miwa]*

The general picture: Cartan matrix of finite type

Feigin-Loktev
fusion product
$\downarrow$

$$|\mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda))|$$

Chari's
$\mathfrak{g}[t]$-modules
$\downarrow$

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| \quad \leq \quad |\mathrm{Hom}_{\mathfrak{g}}(\mathcal{F}^*_{\mathbf{n}}\ ,\ V(\lambda))| \quad \geq \quad M_{\mathbf{n},\lambda}$$

Restricted
$\swarrow$ Sum

$$\vee|$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| \quad = \quad N_{\mathbf{n},\lambda}$$

$\uparrow$
$U_q(\widehat{\mathfrak{g}})$-modules

$\nwarrow$
Unrestricted
Sum

The M-sum formula

For each $\mathbf{n} \in \mathbb{Z}_+^{r\times k}$ and $\lambda \in P^+$ a dominant weight of $\mathfrak{g}$ (simply-laced, rank r), $M_{\mathbf{n},\lambda} := \lim_{k\to\infty} M_{\mathbf{n},\lambda}^{(k)}$ is a **restricted** sum of **non-negative terms:**

$$M_{\lambda,\mathbf{n}}^{(k)} = \sum_{\mathbf{m}\in\mathbb{Z}_+^{r\times k}, p_{\alpha,i}\geq 0} \prod_{\alpha=1}^{r}\prod_{i=1}^{k} \binom{m_{\alpha,i}+p_{\alpha,i}(\mathbf{m})}{m_{\alpha,i}}.$$

$$p_{\alpha,i}(\mathbf{m}) = \sum_j \min(i,j) n_{\alpha,j} - \sum_{\alpha,j} C_{\alpha,\beta}\min(i,j) m_{\beta,j}.$$

The sum is taken over $\mathbf{m} : \sum_i i(n_{\alpha,i} - \sum_\beta C_{\alpha,\beta} m_{\beta,i}) = l_\alpha = h_\alpha(\lambda)$

Counts "string solutions" to Bethe ansatz in generalized Heisenberg model (Kirillov-Reshetikhin)

Known to count the multiplicity in the tensor product of KR-modules in special cases (by bijection of Kirillov, Reshetikhin, Schilling, Shimozono,... from rigged configurations to paths)

The general picture: Cartan matrix of finite type

Feigin-Loktev fusion product ↓

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\,,\;V(\lambda)\right)\right|$$

Chari's $\mathfrak{g}[t]$-modules ↓

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\;V(\lambda)\right)\right| \leq \left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\,,\;V(\lambda)\right)\right| \geq M_{\mathbf{n},\lambda}$$

Restricted Sum ↙ $M_{\mathbf{n},\lambda}$

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\;V(\lambda)\right)\right| \geq \left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\;V(\lambda)\right)\right|$$

$$M_{\mathbf{n},\lambda} = N_{\mathbf{n},\lambda}$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\;V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

↑ $U_q(\widehat{\mathfrak{g}})$-modules

↖ Unrestricted Sum

Theorem (Di Francesco, Kedem)

The equality

$$M_{\mathbf{n},\lambda}^{(k)} = N_{\mathbf{n},\lambda}^{(k)}$$

holds for $\lambda \in P^+$ for any simple Lie algebra if the solutions $Q_{\alpha,m}$ of the Q-system for $\mathfrak{g}$ simple are **polynomials** *in the initial data $\{Q_{\alpha,1}\}_\alpha$ when evaluated at the* **"KR-point"** $\{Q_{\alpha,0} = 1\}_\alpha$.

- The Q-system is a recursion relation for the commutative family $\{Q_{\alpha,m} : 1 \leq \alpha \leq r, m \in \mathbb{Z}\}$. For example:

$$Q_{\alpha,m+1} Q_{\alpha,m-1} = Q_{\alpha,m}^2 - \prod_{\beta \neq \alpha} Q_{\beta,m}^{-C_{\beta,\alpha}}, \quad \text{(when } \mathfrak{g} \text{ simply-laced)}$$

- Theorem: (KR, Nakajima, Hernandez) solutions of the Q-system with $m > 0$ at the KR-point $Q_{\alpha,0} = 1$ are characters of KR-modules restricted to $U_q(\mathfrak{g})$.
- From this it follows that $Q_{\beta,m}$ are polynomials in the variables $\{Q_{\alpha,1}\}_\alpha$.
- This proves that $M = N$.

The general picture: Cartan matrix of finite type

Feigin-Loktev fusion product

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\,,\;V(\lambda)\right)\right|$$

Chari's $\mathfrak{g}[t]$-modules

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\;V(\lambda)\right)\right| = \left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\,,\;V(\lambda)\right)\right| = M_{\mathbf{n},\lambda}$$

Restricted Sum

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\;V(\lambda)\right)\right| = \left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\;V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

$$M_{\mathbf{n},\lambda} \neq N_{\mathbf{n},\lambda}$$

$U_q(\widehat{\mathfrak{g}})$-modules

Unrestricted Sum

Question: Is there a combinatorial explanation of the polynomial property of $Q_{\alpha,m}$ with KR-boundary conditions?

Theorem (K, DFK)

The Q-system for any simple Lie algebra can be formulated as a subgraph of a cluster graph. The initial seed variables are

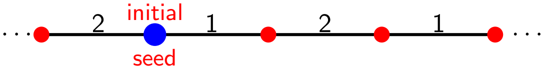
$$\mathbf{x} = (R_{\alpha,0}, R_{\alpha,1})_{\alpha=1,\ldots,r}, \quad B = \begin{pmatrix} C^t - C & -C^t \\ C & 0 \end{pmatrix}$$

(where $R_{\alpha,m} = \epsilon_\alpha Q_{\alpha,m}$ is chosen so that the exchange relation has coefficients equal to 1).

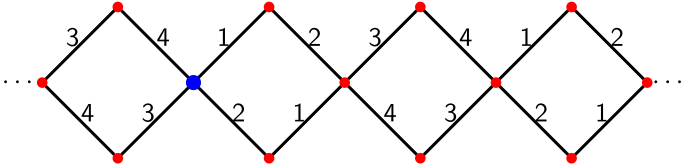
$\Longrightarrow$ Laurent property for solutions of the Q-system.

Examples of cluster graphs

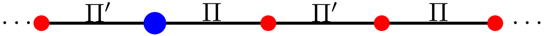
- For $\mathfrak{sl}_2$ the cluster graph is of rank 2:



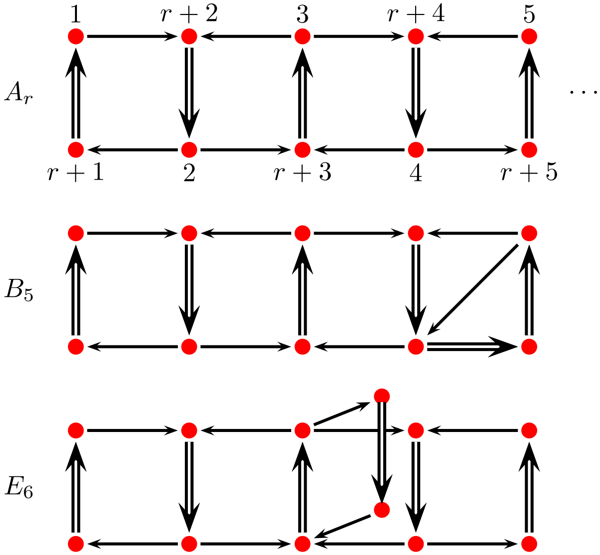
- For $\mathfrak{g} = \mathfrak{sl}_3$ the Q-system subgraph is



- For $\mathfrak{g}$ simply laced define $\mu_\Pi = \mu_1 \circ \cdots \circ \mu_r$ and $\mu_{\Pi'} = \mu_{r+1} \circ \cdots \circ \mu_{2r}$

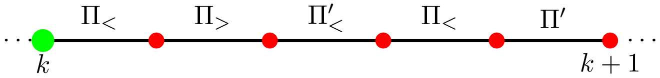


Examples of exchange matrices B

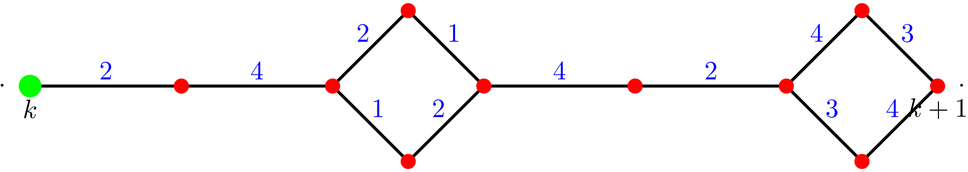


Evolution graphs for non-simply laced cases

F_4, B_r, C_r:

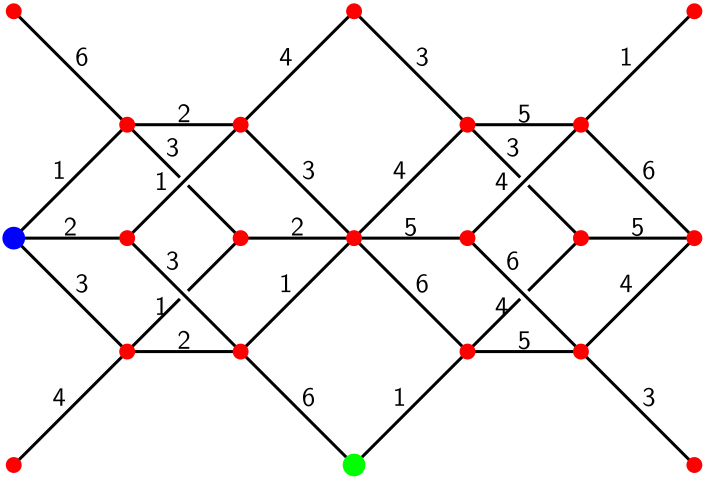


G_2

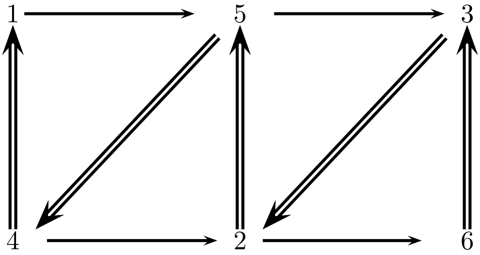


Remarks about cluster graphs and exchange matrices

The graphs above do not include all points in the cluster graph with cluster variables characters. For A_3 all these points have cluster variables which are solutions to the Q-system:



Remark: At the green point in the previous slide, the exchange matrix is



which appears in [Buan, Iyama, Reiten Scott] and [Geiss, Leclerc, Schröer].

Polynomial property at the KR point

- Due to the Laurent property [Fomin-Zelevinsky] we know that $Q_{\alpha,m}$ is a Laurent polynomial in the initial seed variables $\{Q_{\beta,0}, Q_{\beta,1}\}_{\beta=1,\dots,r}$.

Theorem (DFK)

Given the **KR-boundary conditions***, $Q_{\alpha,0} = 1$ for all α,* **all** *cluster variables in any of the the cluster algebras defined above are polynomials in $\{Q_{\alpha,1}\}_\alpha$ at the KR point, because $\{Q_{\alpha,-1} = 0\}$ due to the form of the Q-system.*

- **Remarks:** The cluster algebra is not defined everywhere if the evaluation is done before mutations (no evolution to the left, because $Q_{\alpha,-1} = 0$), but all cluster variables are all well-defined after evaluation at the KR-point due to Laurent property.
- The minus sign normalized away at the beginning of the cluster formulation is useful to keep at this stage, so we drop the cluster variables $R_{\alpha,m}$ and return to $Q_{\alpha,m}$. The Laurent property still holds.

Proof for rank 2: [Fomin] (See Cluster III [BFZ])

$$Q_{m+1}Q_{m-1} = Q_m^2 - 1$$

- We have $B_{2,2} = 0$ and cluster variable $(a, b) = (Q_0, Q_1)$ at the boundary.
- We have $\mu_2(b) = b' = N(a)/b$ where $N(a)$ is a sum of two monomials.
- At the KR point, $N(a) = 0$ by assumption.
- Any cluster variable at any node y looks like a finite sum:

$$y(a, b) = a^{-m} \sum_n C_n(a) b^n = a^{-m} \sum_n C_n(a)(b')^{-n} N(a)^n$$

  where $C_n(a)$ is a polynomial in a.
- If $n < 0$ then due to the Laurent property, $C_n(a)$ is divisible by $N(a)^{|n|}$.
- $C_{n<0}(a) = 0$ after evaluation at the KR point, where $N(a) = 0$.
- $\implies y(a, b)$ is a polynomial in b at the KR point.

Higher rank

For example, the simply-laced case Q-system:

$$Q_{\alpha,m+1}Q_{\alpha,m-1} = Q_{\alpha,m}^2 - \prod_{\beta\neq\alpha} Q_{\beta,m}^{-C_{\beta,\alpha}}$$

- The Theorem follows in all cases from the fact that at the "boundary" of the Q-system $m = 0$, B always has the lower-right $r \times r$ block equal to 0.

$$B = \begin{pmatrix} C^t - C & -C^t \\ C & 0 \end{pmatrix}.$$

- The KR boundary condition always imply that $\mu_i(x_i) = 0$ for $i > r$ at the KR point, where $\mathbf{x}$ is the initial cluster seed.
- The form of the B-matrix implies that the numerator of $\mu_i(x_i)$ with $i > r$ is a function only of x_j with $j \leq r$. So the argument above can be repeated.

Summary

- This gives a purely combinatorial explanation for the polynomiality property – may be applied to the some of the more general Q-systems defined by Kuniba, Nakanishi, Tsuboi (these include T-systems).
- For any system with a "KR point", with the properties that the B-matrix with respect to the variables $(\mathbf{a}, \mathbf{b})$ has a block form with one 0 diagonal block at the “initial” point, the polynomiality property holds (see lemma in DFK).
- In the more general Q-systems there is no $M = N$ identity except in cases corresponding to generalized Cartan matrices.
- Cluster algebra explanation for $M = N$?