Harm Derksen - 3/21/08 2pm - "Quivers with Potentials"

introduced by Fred Goodman

Lecture begins

- joint work with Jerzy Weyman & Andrei Zelevinsky

Quivers with potentials

Quivers = directed graph

$Q = (Q_0, Q_1, h, t)$ where $Q_0 = \{1, 2, \ldots, n\}$ vertices
& Q_1 = arrows

$h, t : Q_1 \longrightarrow Q_0$ head, tail   No Loops!

Connection w/ Clusters (Assume Q has no 2 cycles)

Define a mutation of Q at vertex k:

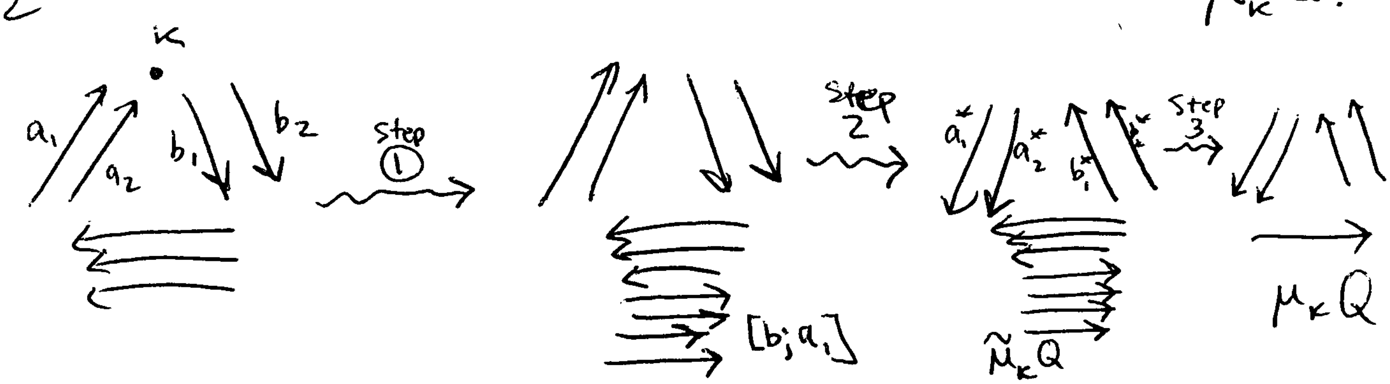
① $\forall\, a: i \to k,\ b: k \to j$, we create $[ba]: i \to j$

② replace every $a: i \to k$ by $a^*: k \to i$
& every $b: k \to j$ by $b^*: j \to k$

③ remove 2-cycles until none are left.

After 3rd step we call the remaining quiver as $\mu_k Q$.
After 2nd " " " " " " $\tilde{\mu}_k Q$.

eg.



Note: $\mu_k^2 = \mathrm{id}$.

Fix $K = \mathbb{C}$. KQ Path algebra is the set of formal lin. comb. of paths in Q. $\forall i \in Q_0$, there is the trivial path $e_i : i \to i$.

$$p \cdot q = \begin{cases} \text{concat } pq & \text{if } tp = hq \\ 0 & \text{otherwise} \end{cases} \qquad 1 = \sum_{i=1}^{n} e_i$$

M fin. dim. KQ-module. $M(i) = e_i M$, $M = \bigoplus_{i=1}^{n} M(i)$.

$a : M \to M$ induces

$a \in Q_1$

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & M \\ M(ta) \arrow[r, "M(a)"'] & M(ha) \arrow[u, hook] \end{tikzcd}$$

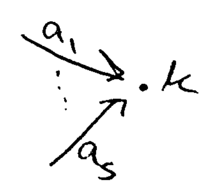
Connection betw: rep^{n} of quivers $\leftrightsquigarrow$ cluster algebras

if Q is acyclic

Goal of Talk: to find this connection for arb. quivers.

BGP - reflection functors if k sink or source.

k sink (no arrow b with $tb = k$)



Define $\mu_k M = \overline{M}$ reps of $\mu_k Q$.

$\overline{M}(i) = M(i)$ if $i \neq k$)

$\overline{M}(b) = M(b)$ if $b \neq a_1 — a_s$

$$M_{in} = \bigoplus_{i=1}^{s} M(ta_i) \xrightarrow{\qquad} M(k)$$
$$\alpha_k = [M(a_1) \; \cdots \; M(a_s)]$$

The kernal is $\begin{bmatrix} M(a_1^*) \\ \vdots \\ M(a_s^*) \end{bmatrix}$ $\quad \overline{M}(k) \longrightarrow M_{in}$.

Now, if k source $\quad \overset{\bullet}{k} \begin{array}{l} \nearrow^{b_1} \\ \vdots \\ \searrow_{b_t} \end{array}$ then we have

$$M(k) \xrightarrow[\beta_k = \begin{bmatrix} M(b_1) \\ \vdots \\ M(b_s) \end{bmatrix}]{} M_{out} = \bigoplus_{j=1}^{t} M(hb_j)$$

$\overline{M}(k)$ is the cokernal of β_k

<u>Remark</u> if α_k is onto, then $\mu_k^2 M \cong M$.

The functors above are <u>only</u> defined if k is a sink or a source.

<u>Quiver with Potentials</u>

$\widehat{KQ}$ is the completion of path algebra (its a topological ring)

A <u>potential</u> is a lin. comb. of cyclic paths of length ≤ 2.

$\quad$ ↳ denoted by S

So a quiver with potential is a pair (Q, S).

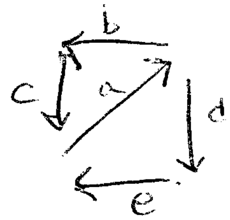
I = closure of span of all $a_\ell a_{\ell-1} \cdots a_1 - a_{\ell-1} \cdots a_1 a_\ell$

$\quad$ where a_ℓ — a_1 cyclic path.

Say $S \underset{cyl}{\sim} S'$ if $S - S' \in I$

$a_\ell a_{\ell-1} \cdots a_1$ cyclic path, then

$$\frac{\partial}{\partial b} a_\ell a_{\ell-1} \cdots a_1 = \sum_{\substack{i \\ a_i = b}} a_{i-1} \cdots a_1 a_\ell \cdots a_{i+1}$$

eg.  $\frac{\partial}{\partial a} dacbae = cbaed + edacb$

$J(S)$ = closure of ideal generated by $\frac{\partial}{\partial a} S$, $a \in Q$.

$P(Q,S) = KQ/J(S)$ is called the Jacobian algebra.

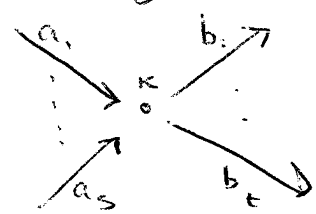
How do we mutate Quivers with Potentials?

We did this with Q (see page 1). But how about for (Q,S)?

$\tilde{S}$ potential for $\tilde{\mu}_k Q$. $\tilde{S} = [S] + \Delta$

$[S] =$ S where $b_j a_i$ is replaced by $[b_j a_i]$.

$\Delta = \sum_{i,j} [b_j a_i] a_i^* b_j^*$



Claim: there exists an automorphism $\varphi: \widehat{K\tilde{Q}} \xrightarrow{\sim} \widehat{K\tilde{Q}}$

s.t. $\varphi(\tilde{S}) = \sum_{i=1}^{N} u_i v_i + \bar{S}$ where $u_1 - u_N$, $v_1 - v_N$ distinct arrows not appearing in $\bar{S}$. The $u_i v_i$ are 2-cycles.

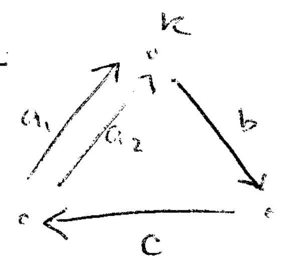
Then $\bar{Q} = \tilde{Q} \setminus \{u_1 - u_N, v_1 - v_N\}$. If S is "generic enough", then $\bar{Q}$ has no 2-cycles. $\bar{Q} = \mu_k Q$.

Remark: $\bar{S}$ has no 2 cycles.

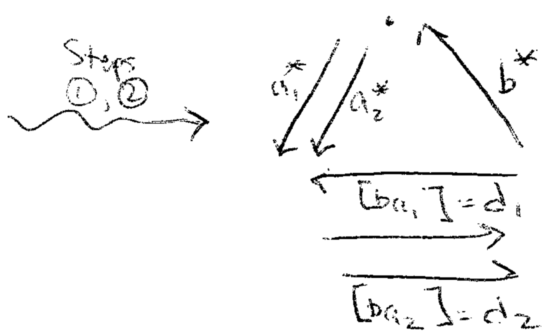
<u>Define</u> $\mu_k(Q,S) = (\overline{Q}, \overline{S})$. We say $(Q,S) \sim_{\text{right}} (Q',S')$ are <u>right equivalent</u> if $\exists$ isom $\varphi: \widehat{KQ} \xrightarrow{\sim} \widehat{KQ'}$ s.t. $\varphi(S) \sim_{\text{cycl}} S'$. This implies $P(Q,S) \cong P(Q',S')$.

So our def$^{\underline{n}}$ above for $\mu_k(Q,S)$ is well-defined up to right equivalence.

<u>Example:</u>

$Q =$ 

$S = cba_1$



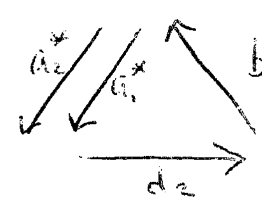
is $\widetilde{Q} = \widetilde{\mu}_k Q$

$$\widetilde{S} = \underbrace{cd_1}_{[S]} + \underbrace{d_1 a_1^* b^* + d_2 a_2^* b}_{\triangle}$$

Now we need to get rid of 2 cycles. Apply φ where $\varphi(c) = c - a_1^* b^*$ & that will cancel some terms in $\widehat{S}$.

$\varphi(\widetilde{S}) \sim_{\text{cycl}} cd_1 + \underbrace{d_2 a_2^* b^*}$ $\to$ This will become my $\overline{S}$.

So $\mu_k Q = \overline{Q} = \widetilde{Q} \setminus \{c, d_1\}$



$\overline{S} = d_2 a_2^* b^*$

$\mu_k(Q,S) = (\overline{Q}, \overline{S})$

<u>Def$^{\underline{n}}$</u> A rep$^{\underline{n}}$ of (Q,S) is a pair $\mathcal{M} = (M, V)$ where M is a $P(Q,S)$-module $V = (V_1, \ldots, V_n)$ V_i = fin. dim'l vect. spaces = "decorations"

<u>The Challenge</u>: to define $\mu_k \mathcal{M}$.

When M is a sink, we had $M_{in} \xrightarrow{\alpha_k} M(k)$ & when M source $M(k) \xrightarrow{\beta_k} M_{out}$

& we have a 3rd map giving

$$\begin{tikzcd} & M(k) \arrow[dr, "\beta_k"] & \\ M_{in} \arrow[ur, "\alpha_k"] & & M_{out} \arrow[ll, bend left, "\gamma_k = \gamma_k(S)"] \end{tikzcd}$$

$\alpha_k \beta_k = 0$

$\alpha_k \gamma_k = 0$

$$\overline{M}(k) = {\ker \alpha_k}/{\operatorname{im} \gamma_k} \oplus \operatorname{im} \gamma_k \oplus {\ker \gamma_k}/{\operatorname{im} \beta_k} \oplus V_k$$

<u>Given</u> a Quiver Q, you can assoc. a matrix $B = B(Q) = (b_{ij})_{1 \le i, j \le n}$

where $b_{ij} = (\#\text{ arrows } j \longrightarrow i) - (\#\text{ arrows } i \to j)$

Observe, $B(\mu_k Q) = \mu_k B$. $\mathcal{M} = (M, V)$ QP-rep's.

$d = (d_1, \dots, d_n)$ $d_i \in \mathbb{N}$. $Gr_d(\mathcal{M})$ = variety of d-dim'l subrep's of M

$$\begin{gathered} Gr_d(\mathcal{M}) \\ \cap\!\! \mid \\ \prod_i Gr_{d_i}(M(i)) \end{gathered} \qquad K = \mathbb{C}$$

$\chi(Gr_d(\mathcal{M}))$ Euler Characteristic.

$$F_{\mathcal{M}}(y_1 - y_n) = \sum_d \chi(Gr_d(\mathcal{M}))\, y^d \qquad y^d = \prod y_i^{d_i}$$

$$h_{\mathcal{M}} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \quad \& \quad g_{\mathcal{M}} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \qquad \begin{aligned} h_i &= -\dim \ker \beta_i \\ g_i &= \dim \gamma_i - \dim M_i + \dim V_i \end{aligned}$$

$$\overline{\mathcal{M}} = \mu_k \mathcal{M} \qquad (1+y_k)^{h_k} F_{\mathcal{M}}(y_1, \dots, y_n) = (1+\bar{y}_k)^{\bar{h}_k} F_{\overline{\mathcal{M}}}(\bar{y}_1 \dots \bar{y}_n)$$

where $$y_i = \begin{cases} y_k^{-1} & \text{if } i = k \\ y_i\, y_k^{[-b_{ki}]_+} (y_k+1)^{b_{ki}} \end{cases} \qquad g_k = h_k - \bar{h}_k$$

$S_k^- = (0, V)$ $\qquad$ $V_i = \begin{cases} 0, & \text{if } i \neq k \\ k & \text{if } i = k \end{cases}$

$(M, V) = \mu_{i_d} \cdots \mu_{i_1} S_k^-$

$F_M(y_1 \text{———} y_N) \qquad g_M$

$\boxed{\text{End of Lecture}}$

Let us thank the speaker ☺