Reduced standard modules, filtrations, and cohomology

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Joint work with Ed Cline and Leonard Scott

If G is a simply connected, semisimple group over an algebraically closed field k of characteristic $p > 0$, the standard (Weyl) modules

$$\Delta(\lambda), \quad \lambda \in X^+$$

and the costandard modules

$$\nabla(\lambda), \quad \lambda \in X^+$$

are obtained by reduction mod p from the characteristic zero irreducible representation $L_{\mathbb{C}}(\lambda)$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ using minimal and maximal lattices, respectively.

These modules sometimes behave like the irreducible modules in characteristic zero, e. g.,

$$\dim \operatorname{Ext}^n_G(\Delta(\lambda), \nabla(\mu)) = \delta_{n,0}\delta_{\lambda,\mu}.$$

Additionally, they have characters given by Weyl's character formula.

Let U_ζ be the quantum enveloping algebra at a pth root of unity ζ. Let $\Delta^{\mathrm{red}}(\lambda)$ (resp., $\nabla_{\mathrm{red}}(\lambda)$ be the rational G-module obtained by 'reduction mod p' from irreducible U_ζ-module $L_\zeta(\lambda)$. Here we work over an appropriate p-modular system $(\mathcal{O}, K, k)$. These modules have been defined by Lusztig.

<u>Elementary properties:</u>

(1) $\Delta^{\mathrm{red}}(\lambda)$ and $\nabla_{\mathrm{red}}(\lambda)$ are indecomposable;
(2) $\Delta(\lambda) \twoheadrightarrow \Delta^{\mathrm{red}}(\lambda)$ and $\nabla_{\mathrm{red}}(\lambda) \hookrightarrow \nabla(\lambda)$.
(3) head $\Delta^{\mathrm{red}}(\lambda) \cong \operatorname{soc} \nabla_{\mathrm{red}}(\lambda) \cong L(\lambda)$.
(4) Assume the LCF holds for U_ζ. Then

$$\mathrm{ch}\ \Delta^{\mathrm{red}}(\lambda) = \mathrm{ch}\ \nabla_{\mathrm{red}}(\lambda) = \chi_{KL}(\lambda).$$

We are interested in the homological properties of these modules. Often we assume that $p > h$ (the Coxeter number) and that the Lusztig conjecture holds for G.

Let
$$X_1^+ = \{\lambda \in X^+ \,|\, (\lambda + \rho, \alpha_i) \le p, \alpha_i^\vee \in \Pi\}$$
be the restricted dominant weights.

Proposition: (Z. Lin) For $\lambda \in X^+$, write
$$\lambda = \lambda_0 + p\lambda_1, \quad \text{where}\, \lambda_0 \in X_1^+, \lambda_1 \in X^+.$$
Then
$$\Delta^{\mathrm{red}}(\lambda) = \Delta^{\mathrm{red}}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$$
and
$$\nabla_{\mathrm{red}}(\lambda) = \nabla_{\mathrm{red}}(\lambda_0) \otimes \nabla(\lambda_1)^{(1)}.$$

A proof can be based on the following unpublished result:

Lemma: (E. Cline) For a restricted weight $\lambda \in X_1^+$, there is a surjective G_1B-morphism
$$\widehat{Z}_1(\lambda) \twoheadrightarrow \Delta(\lambda).$$

Therefore, if p is large enough that the regular restricted weights are contained in the Janzten region, namely, if

$$p \geq 2h - 3,$$

and if the Lusztig conjectue holds, then

$$\Delta^{\mathrm{red}}(\lambda) = L(\lambda_0) \otimes \Delta(\lambda_1)^{(1)},$$

and

$$\nabla_{\mathrm{red}}(\lambda) = L(\lambda_0) \otimes \nabla(\lambda_1)^{(1)}.$$

Sometimes it is useful to set

$$\Delta^{\mathrm{red}}(\lambda)' := L(\lambda_0) \otimes \Delta(\lambda_1)^{(1)},$$
$$\nabla_{\mathrm{red}}(\lambda)' := L(\lambda_0) \otimes \nabla(\lambda_1)^{(1)}.$$

In any event, for $\lambda \in X^+$, $\Delta^{\rm red}(p\lambda) \cong \Delta(\lambda)^{(1)}$ and $\nabla_{\rm red}(p\lambda) = \nabla(\lambda)^{(1)}$. For $x, y \in W_p$, let $P_{y,x}$ = corresponding Kazhdan-Lusztig polynomial.We have the following result, calculating the groups $\mathrm{Ext}^\bullet_G(\Delta(\lambda)^{(1)}, \nabla(\mu))$:

Proposition: Assume that $p > h$. For $\lambda \in X^+$, write $p\lambda = x \cdot \tau^-$, $x \in W_p$ and $\tau^- \in C^-_{\mathbb{Z}}$ (= p-alcove containing -2ρ). Then:

$$t^{l(x)-l(y)}\overline{P}_{y,x} = \sum_{n=0}^{\infty} \dim\ \mathrm{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(y\cdot\tau^-))t^n$$

for $y \in W_p$ s.t. $y \cdot \tau^- \in X^+$.

Also,

$$\mu(y,x) = \dim\ \mathrm{Ext}^1_G(\Delta(\lambda)^{(1)}, \nabla(y\cdot\tau^-)) \leq 1,$$

where $\mu(y,x)$ = coef. of $t^{l(x)-l(y)-1}$ in $P_{y,x}$.

This result is proved using the Kumar-Lauritzen-Thomsen (based on Andersen-Janzten) results mentioned in Pillen's talk. An ingredient here was the Kostant partition function $\mathfrak{p}_n$. The last sentence comes about because for $n = 0$,

$$\mathfrak{p}_0(\sigma) = \delta_{\sigma,0},$$

for all $\sigma \in X$. !!

Enriched Grothendieck groups K_0^L and K_0^R

Both K_0^L and K_0^R are free modules over $\mathbb{Z}[t, t^{-1}]$, each with a basis indexed by the set X^+ of dominant weights. These bases are paired to each other with respect to a natural sesquilinear pairing

$$\langle \, , \, \rangle : K_0^L \times K_0^R \to \mathbb{Z}[t, t^{-1}].$$

The groups K_0^L and K_0^R can be viewed as deformations of $K_0(D^b(G\text{–mod})) = K_0(G\text{–mod})$, which preserve homological degree information by tracking parity conditions. If $M, N \in G$–mod are "represented" in K_0^L and K_0^R, respectively—say by $[M]$ and $[N]$—then

$$\sum_i \dim \operatorname{Ext}^i_G(M, N) t^i = \langle [M], [N] \rangle.$$

Thus, if $[M], [N]$ can be expressed in terms of the paired bases above, $\operatorname{Ext}^\bullet_G(M, N)$ is calculated.

More precisely, $M \in D^b(G\text{–mod})$ is represented in K_0^L when M or its shift $M[1]$ belongs to $\mathcal{E}^L = \mathcal{E}^L(G\text{–mod})$, and direct sums of such objects—those in $\mathcal{E}^L \oplus \mathcal{E}^L[1]$—are also represented in K_0^L. Here $\mathcal{E}^L$ is a full subcategory of $D^b(G\text{–mod})$ roughly consisting of those objects which have a "filtration" with "sections" of the form $\Delta(\lambda)[i]$, $\lambda \in X^+$, $i \equiv l(\lambda) \bmod 2$, for a suitable "length" function $l : X^+ \to \mathbb{N}$. A similar description applies to K_0^R, involving another full subcategory $\mathcal{E}^R$ defined in terms of the modules $\nabla(\lambda)$.

To make further progress on the homological properties of the Δ^{red} and ∇_{red}-modules, we need:

Definition: The left (resp., right) *homological lattice property* hLP^L (resp., hLP^R) holds for $\lambda \in X^+_{\text{reg}}$ if $L_\zeta(\lambda)$ has an admissible lattice $\widetilde{L}_\zeta(\lambda)$ (resp., $\widetilde{L}'_\zeta(\lambda)$) s.t.

$$\text{Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu))$$

(resp.,

$$\text{Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{\Delta}_\zeta(\mu), \widetilde{L}'_\zeta(\lambda)))$$

is $\mathcal{O}$-torsion-free when $\mu \leq \lambda$.

Theorem: Assume that $p > h$.

(a) For $\lambda \in X^+_{\text{reg}}$, hLP^L (resp. hLP^R) holds if and only if $\Delta^{\text{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^L$ (resp., $\nabla_{\text{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^R$).

(b) If hLP^L (resp., hLP^R) holds for $\lambda, \mu \in X^+_{\mathrm{reg}}$, then

$$\dim \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}(\lambda), \nabla_{\mathrm{red}}(\mu)) = \sum_{m=0}^{n} \sum_{\nu} \dim \mathrm{Ext}^m_G(\Delta^{\mathrm{red}}(\lambda), \nabla(\nu)) \times \dim \mathrm{Ext}^{n-m}_G(\Delta(\nu), \nabla_{\mathrm{red}}(\mu)).$$

Furthermore, if $\lambda = x \cdot \lambda^-$, $\lambda^- \in C^-_{\mathbb{Z}}$, then

$$t^{l(x)-l(y)}\overline{P}_{y,x} = = \sum_{n=0}^{\infty} \dim\ \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}(\lambda), \nabla(y \cdot \lambda^-))t^n = \sum_{n=0}^{\infty} \dim\ \mathrm{Ext}^n_G(\Delta(y \cdot \lambda^-), \nabla_{\mathrm{red}}(\lambda))t^n.$$

In particular,

$$\dim \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}(\lambda), \nabla_{\mathrm{red}}(\mu)) = \dim\ \mathrm{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\mu)).$$

Four Conjectures, $p > h$

Conjecture 1. hLP^L, hLP^R hold $\forall \lambda \in X^+_{\mathrm{reg}}$.

Conjecture 2. For $\lambda \in X^+_{\mathrm{reg}}$, $\Delta^{\mathrm{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^L$ and $\nabla_{\mathrm{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^R$ ($\iff$ $\Delta^{\mathrm{red}}(\lambda) \in \widehat{\mathcal{E}^L}$ and $\nabla_{\mathrm{red}}(\lambda) \in \widehat{\mathcal{E}^R}$.)

Conjecture 3. For $\lambda \in X^+_{\mathrm{reg}}$, write $\lambda = \lambda_0 + p\lambda_1$, with $\lambda_0 \in X^+_1$ and $\lambda_1 \in X^+$. Then $\Delta^{\mathrm{red}}(\lambda)'[-l(\lambda)] \in \mathcal{E}^L$ and $\nabla_{\mathrm{red}}(\lambda)'[-l(\lambda)] \in \mathcal{E}^R$.

Conjecture 4. For $\lambda \in X^+$, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) has a Δ^{red}-filtration, i. e., a filtration as a G-module with sections of the form $\Delta^{\mathrm{red}}(\nu)$ (resp., $\nabla_{\mathrm{red}}(\nu)$), $\nu \in X^+$.

Theorem: Let $p > h$.

(a) Assume that the LCF holds for all regular weights in X_1^+. Then Conjectures 1,2, and 3 are true.

(b) If Conjecture 3 holds and $p \geq 2h - 3$, then the LCF holds for all regular weights in X_1^+.

For $\lambda \in X^+_{\mathrm{reg}}$, put

$$E_\zeta(\lambda) = \Delta_\zeta(\lambda)/\mathrm{rad}^2\Delta_\zeta(\lambda).$$

Let $\widetilde{E}(\lambda)$ be the image of $\widetilde{\Delta}_\zeta(\lambda)$ in $E_\zeta(\lambda)$, and set $E(\lambda) = \widetilde{E}(\lambda)/\pi\widetilde{E}(\lambda)$. Here $(\pi) = \mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Observe that $\widetilde{E}(\lambda)$ is an admissible lattice for $E_\zeta(\lambda)$.

Theorem: Assume LCF holds for all regular weights in X_1^+. Let

$$\widetilde{D}(\lambda) = \mathrm{Ker}(\widetilde{E}(\lambda) \twoheadrightarrow \widetilde{L}^{\min}(\lambda)),$$

and

$$D_\zeta(\lambda) = \mathrm{Ker}(E_\zeta(\lambda) \twoheadrightarrow L_\zeta(\lambda)).$$

Then $\widetilde{D}(\lambda)$ has a $\widetilde{U}_\zeta$-filtration with (distinct) sections $\widetilde{L}^{\min}(\mu)^{\oplus n_\mu}$, where

$$n_\mu = [D_\zeta(\lambda) : L_\zeta(\mu)] = \dim \mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu)).$$

In particular, $E(\lambda)$ has a Δ^{red}-filtration with top section $\Delta^{\mathrm{red}}(\lambda)$.

At the start of this talk, the standard modules $\Delta(\lambda)$ and the costandard modules $\nabla(\lambda)$ were introduced. Of course, they form the standard and costandard modules for the highest weight category of rational G-modules. So, it very natural to ask of the reduced standard modules $\Delta^{\mathrm{red}}(\lambda)$ and reduced costandard modules $\nabla_{\mathrm{red}}(\lambda)$ are the standard modules and costandard modules for a highest weight category in some natural way.

For a regular dominant weight λ, write

$$\lambda = w \cdot \lambda^-,$$

where $\lambda^- \in C^-_{\mathbb{Z}}$. Define $l(\lambda) = l(w)$.

Say that $L(\lambda)$ has **even** (resp., **odd**) **parity** if $l(\lambda) \equiv 0$ (resp., $l(\lambda) \not\equiv 0$) mod 2. Let

$$X^+_{\text{reg, even}} \quad (\text{resp., } X^+_{\text{reg,odd}})$$

be the set of regular dominant weights of even (resp., odd) parity. Let $\mathcal{C}^{\text{reg}}_{\text{even}}$ (resp., $\mathcal{C}^{\text{reg}}_{\text{odd}}$) be the full subcategory of $\mathcal{C}$ (the category of rational G-modules) generated by the irreducible modules $L(\lambda)$ for $\lambda \in X^+_{\text{reg, even}}$ (resp., $\lambda \in X^+_{\text{reg,odd}}$).

Theorem: (PS) Assume that $p > h$ and that the LCF holds for all regular restricted dominant weights. Then $\mathcal{C}^{\text{reg}}_{\text{even}}$ is a highest weight category with weight poset $X^+_{\text{reg, even}}$. For $\lambda \in X^+_{\text{reg, even}}$, the corresponding standard (resp., costandard) object is $\Delta^{\text{red}}(\lambda)$ (resp., $\nabla_{\text{red}}(\lambda)$).

Similarly, $\mathcal{C}^{\text{reg}}_{\text{odd}}$ is a highest weight category with weight poset $X^+_{\text{reg,odd}}$. For $\lambda \in X^+_{\text{reg,odd}}$, the corresponding standard (resp., costandard) object is $\Delta^{\text{red}}(\lambda)$ (resp., $\nabla_{\text{red}}(\lambda)$.

Let

$$i_*^+ : \mathcal{C}_{\text{even}}^{\text{reg}} \to \mathcal{C} \ (\text{resp.}, i_*^- : \mathcal{C}_{\text{odd}}^{\text{reg}} \to \mathcal{C})$$

denote the canonical full embedding of categories. The functor i_*^+ is exact, and admits a right adjoint $i_+^! : \mathcal{C} \to \mathcal{C}_{\text{even}}^{\text{reg}}$ and a left adjoint $i_+^* : \mathcal{C} \to \mathcal{C}_{\text{even}}^{\text{reg}}$.

Explicitly, for $M \in \mathcal{C}$, $i_+^! M$ (resp., $i_+^* M$) is the largest submodule (resp., quotient module) of M having composition factors $L(\lambda)$ with $\lambda \in X_{\text{reg, even}}^+$.

Similarly, i_*^- admits a right and left adjoints $i_-^!$ and i_-^*.

For $\lambda \in X_{\text{reg}}^+$, define $X^{\text{red}}(\lambda)$ by means of the following exact sequence

$$0 \to X^{\text{red}}(\lambda) \to \Delta(\lambda) \to \Delta^{\text{red}}(\lambda) \to 0.$$

Also, define $Y_{\text{red}}(\lambda)$ by the following exact sequence

$$0 \to \nabla_{\text{red}}(\lambda) \to \nabla(\lambda) \to Y_{\text{red}}(\lambda) \to 0.$$

Theorem. (PS) Assume that $p > h$ and that the LCF holds for all regular restricted dominant weights. Let $\lambda \in X^+_{\text{reg, even}}$. Then

$$i^*_- X^{\text{red}}(\lambda) \text{ (resp., } i^!_- Y_{\text{red}}(\lambda))$$

has a Δ^{red}-filtration (resp., ∇_{red}-filtration) in $\mathcal{C}^{\text{reg}}_{\text{odd}}$.

Observe that $i^*_- X^{\text{red}}(\lambda)$ is the largest quotient of $X^{\text{red}}(\lambda)$ whose composition factors are all odd.

Proof: One shows that

$$\operatorname{Ext}^1_{\mathcal{C}^{\text{reg}}_{\text{odd}}}(i^*_- X^{\text{red}}(\lambda), \nabla_{\text{red}}(\mu)) = 0$$

for all $\mu \in X^+_{\text{reg,odd}}$. Q.E.D.

For $\lambda \in X^+_{\text{reg, even}}$, recall

$$E_\zeta(\lambda) = \Delta_\zeta(\lambda)/\text{rad}^2\Delta_\zeta(\lambda).$$

Let $\widetilde{E}(\lambda)$ be be a minimal lattice in $E_\zeta(\lambda)$, and let $E(\lambda)$ be its reduction mod p. Let

$$D(\lambda) = \text{Ker}(E(\lambda) \twoheadrightarrow \Delta^{\text{red}}(\lambda)).$$

Theorem. (PS) Let $\lambda \in X^+_{\text{reg, even}}$. Then

$$D(\lambda) \cong i^*_- X^{\text{red}}(\lambda)$$

and

$$i^*_+\Delta(\lambda) \cong \Delta^{\text{red}}(\lambda).$$

We also have the provisional result:

Theorem: (PS, 2008) We have

$$E(\lambda) \cong \Delta(\lambda)/\operatorname{Rad}^2_{G_1T}\Delta(\lambda)$$

$$D(\lambda) \cong \operatorname{Rad}^1_{G_1T}(\Delta(\lambda))/\operatorname{Rad}^2_{G_1T}(\Delta(\lambda)).$$

Applications and Further Directions

Given $\lambda \in X^+$, write $\lambda = \sum_{i=0}^{\infty} p^i \lambda_i$, where $\lambda_i \in X_1^+$. We make no assumption on p, except those explicitly noted below. Put

$$\lambda^{(i)} = \sum_{j=i}^{\infty} p^{j-i} \lambda_j.$$

Theorem: (PS) Assume that $p > h$ and that the LCF holds for all regular weights in X_1^+. Let $\lambda, \mu \in X^+$ be distinct weights with $\lambda > \mu$ and let j minimal with $\lambda_j \neq \mu_j$. Suppose that $\lambda^{(j)} \in X_{\text{reg}}^+$. Then

$$\begin{aligned} &\dim \operatorname{Ext}^1_G(L(\lambda), L(\mu)) \\ &\leq \dim \operatorname{Ext}^1_{U_\zeta}(L_\zeta(\lambda^{(j)}), L_\zeta(\mu^{(j)})). \end{aligned}$$

The last term can in principle be calculated, suggesting a way to bound 1-cohomology. More precisely,

$\dim\ \mathrm{Ext}^1_{U_\zeta}(L_\zeta(y\cdot(-2\rho)), L_\zeta(w\cdot(-2\rho))) = \mu(y, w)$

is independent of p, if $p > h$. Such a bound is easy to find for each prime p. In fact, $\dim\ \mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu))$ is at most the dimension of the μ-weight space in $\mathrm{St}_\zeta \otimes \mathrm{St}_\zeta \otimes L_\zeta(\lambda)$. This multiplicity is at most

$$\mathfrak{p}(2(p-1)\rho).$$

Theorem. (CPS) There is a constant C, depending only on Φ such that: Let the semisimple group G be defined over an algebraically closed field k of characteristic p, and let $\sigma : G \to G$ be an endomorphism such that G_σ is a finite group. Then for all irreducible L:

$$\dim\ H^1(G_\sigma, L) \leq C.$$

Remarks: (1) At the end of this argument, key use is made of Bendel-Nakano-Pillen work on the module $\mathcal{G}_r(k)$ discussed in Pillen's talk yesterday.

(2) The Ree and Suzuki group case is due to P. Sin.

We also mention the curious fact, whose proof depends on the inequality
$\mu(y, x) = \dim\ \mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(p\lambda), \nabla(y{\cdot}\tau^-)) \leq 1,$
mentioned earlier. (Here $p\lambda = x \cdot \tau^-$.)

Theorem: (CPS) Assume that $p > h$. Let $\lambda = \tau + p\nu \in X^+$, with $0 \neq \tau \in X_1^+$ and $\nu \in X^+$. Suppose that $\dim\ H^1(G, L(\lambda)) > 1$. Then $\tau > p\nu^\star$, where $\nu^\star := -w_0(\nu) \in X^+$.