Control of fusion in saturated fusion systems

Radha Kessar
University of Aberdeen

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Outline

Saturated fusion systems

Local subsystems

Control of fusion

Nilpotency Criteria

Control of Transfer

Notation

For Y, Z subgroups of a group X,

$$\mathrm{Hom}_X(Y, Z) := \{\phi : Y \to Z \ : \ \exists x \in X \text{ s.t. } \phi(y) = xyx^{-1} \forall y \in Y\}$$

$$\mathrm{Aut}_X(Y) := \mathrm{Hom}_X(Y, Y) \cong N_X(Y)/C_X(Y)$$

$$Out_X(Y) := \mathrm{Aut}_X(Y)/\mathrm{Aut}_Y(Y) \cong N_X(Y)/YC_X(Y)$$

Categories on p-groups

Definition

Let p be a prime number and P a finite p-group. A category on P is a category $\mathcal{F}$ with
Objects : subgroups of P
Morphisms: For $Q, R \leq P$, $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ consists of injective group homomorphisms from Q to R;

- *Composition is the usual composition of maps;*
- *If $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, then the induced isomorphism $Q \cong \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.*

Example

G a finite group, P a Sylow p-subgroup of G. The category $\mathcal{F}_P(G)$ on P with $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \mathrm{Hom}_G(Q, R)$ for $Q, R \leq P$.

Let P be a finite p-group, $\mathcal{F}$ a category on P and $Q, R \leq P$.

Definition

- *R is fully $\mathcal{F}$-centralized if $|C_P(R)| \geq |C_P(R')|$ for any $R' \leq P$ which is $\mathcal{F}$-isomorphic to R.*
- *R is fully $\mathcal{F}$-normalized if $|N_P(R)| \geq |N_P(R')|$ for any $R' \leq P$ which is $\mathcal{F}$-isomorphic to R.*
- *For a group isomorphism $\phi : Q \to R$, N_ϕ:= inverse image of $\phi^{-1}\mathrm{Aut}_P(R)\phi \cap \mathrm{Aut}_P(Q)$ under the map*

$$N_P(Q) \to \mathrm{Aut}_P(Q), \quad x \to (y \to xyx^{-1}).$$

- $QC_P(Q) \leq N_\phi$.

Example

$\mathcal{F} = \mathcal{F}_P(G)$.

- $R, R' \leq P$ are $\mathcal{F}$-isomorphic iff $R' = gRg^{-1}$ for some g in G.
- R fully $\mathcal{F}$-centralized iff $C_P(R)$ is a Sylow p-subgroup of $C_G(R)$ and R fully $\mathcal{F}$-normalized iff $N_P(R)$ is a Sylow p-subgroup of $C_G(R)$.
- If $\phi : Q \to R$ is an isomorphism induced by $g \in G$, then $N_\phi = g^{-1}(N_P(R)C_G(R))g \cap N_P(Q)$.

Definition

(Puig, '90) Let P be a finite p-group. A saturated fusion system on P is a category $\mathcal{F}$ on P such that

- $\mathrm{Hom}_P(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R)$ *for all* $Q,R \leq P$.
- $Aut_P(P)$ *is a Sylow p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(P)$. *(Sylow Axiom)*
- *If $R \leq P$ is fully $\mathcal{F}$-normalized then any $\mathcal{F}$-isomorphism $\phi : Q \to R$ extends to a $\mathcal{F}$-morphism $\hat{\phi} : N_\phi \to P$. (Extension axiom)*

Proposition

G a finite group, P a Sylow p-subgroup of G. Then $\mathcal{F}_P(G)$ is a saturated fusion system on P.

- Fusion systems play a role in
  - Finite group theory
  - Modular representation theory
  - Homotopy theory

- There exist *exotic* saturated fusion systems, that is, saturated fusion systems which do not come from finite groups, (nor from p-blocks of finite groups), but there is no systematic way to construct these.
- Each known exotic fusion system has a unique p-local finite group associated to it, but the general problem of existence and uniquness is open (obstruction theory of Broto-Levi-Oliver).
- Robinson, Leary-Stancu: Each saturated fusion system occurs as the fusion system of a (possibly) infinite group.
- Open question: Does every fusion system of a p-block of a finite group occur as the fusion system of some (possibly other) finite group?

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Local Subsystems

Let $\mathcal{F}$ be a saturated fusion system on P and let $Q \leq P$.

Definition

$C_{\mathcal{F}}(Q)$:= category on $C_P(Q)$ defined by:
For $R, R' \leq C_P(Q)$, the morphism set in $C_{\mathcal{F}}(Q)$ from R to R' consists of all group homomorphisms $\varphi : R \to R'$ such that there exists a morphism $\psi : RQ \to R'Q$ in $\mathcal{F}$ satisfying $\psi|_R = \varphi$ and $\psi|_Q = Id_Q$.

$N_{\mathcal{F}}(Q)$:= category on $N_P(Q)$ defined by:
For $R, R' \leq N_P(Q)$, the morphism set in $N_{\mathcal{F}}(Q)$ from R to R' consists of all group homomorphisms $\varphi : R \to R'$ such that there exists a morphism $\psi : RQ \to R'Q$ in $\mathcal{F}$ satisfying $\psi|_R = \varphi$ and $\psi(Q) = Q$.

Proposition. (Puig) If Q is fully $\mathcal{F}$-centralised then $C_{\mathcal{F}}(Q)$ is a saturated fusion system on $C_P(Q)$. If Q is fully $\mathcal{F}$-normalised then $N_{\mathcal{F}}(Q)$ is a saturated fusion system on $N_P(Q)$.

Example

$\mathcal{F} = \mathcal{F}_P(G)$.

- $C_{\mathcal{F}}(Q) = \mathcal{F}_{C_P(Q)}(C_G(Q))$.
- $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(N_G(Q))$.

$\mathcal{F}$ saturated fusion system on P, $Q \leq P$.

Definition

- *Q is $\mathcal{F}$-centric if $C_P(Q') = Z(Q')$ for all $Q' \leq P$ such that Q' is $\mathcal{F}$-isomorphic to Q.*
- *Q is $\mathcal{F}$-radical if $\mathrm{Aut}_Q(Q)$ is the largest normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.*

Example

$\mathcal{F} = \mathcal{F}_P(G)$.

- *Q is $\mathcal{F}$-centric if and only if $C_G(Q) = Z(Q) \times C$, for some p'-group C.*
- *Q is $\mathcal{F}$-radical if and only if $O_p(N_G(Q)/QC_G(Q)) = 1$.*

Theorem

(Alperin's fusion theorem) As category on P, $\mathcal{F}$ is generated by $\{\mathrm{Aut}_{\mathcal{F}}(Q) \ : \ Q \le P, \ \mathcal{F}-$ *centric, radical, fully normalized* $\}$

Theorem

(Broto-Castellana-Grodal-Levi-Oliver,'05; for blocks-Külshammer-Puig, '90.)
Let Q be an $\mathcal{F}$-centric, fully $\mathcal{F}$-normalised subgroup of P. Then, there is upto isomorphism, a unique finite group $L = L_Q^{\mathcal{F}}$ having $N_P(Q)$ as Sylow p-subgroup such that $C_L(Q) = Z(Q)$, Q is normal in L and $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(L)$.

Notation. The group $L_Q^{\mathcal{F}}$ will be referred to as the BCGLO-model for $\mathcal{F}$ at Q.

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Control of fusion

Let $\mathcal{N} \leq \mathcal{F}$ be saturated fusion systems on P.

Definition

$\mathcal{N}$ controls fusion in $\mathcal{F}$ if $\mathcal{N} = \mathcal{F}$.

If $\mathcal{F} = \mathcal{F}_P(G)$, $P \leq N \leq G$, $\mathcal{N} = \mathcal{F}_P(N)$, then
$\mathcal{F} = \mathcal{N} \iff \mathrm{Hom}_G(Q, R) = \mathrm{Hom}_N(Q, R)$ for all $Q, R \leq P \iff$
For all $Q, R \leq P$, $g \in G$ such that $gQg^{-1} \leq R$, there exists $n \in N$, $c \in C_G(Q)$ such that $g = nc$, i.e. N controls (strong) p-fusion in G.

Theorem

(Burnside) Let p be a prime, G a finite group and P a Sylow p-subgroup of G. If P is abelian then $N_G(P)$ controls p-fusion in G.

Theorem

Suppose that $\mathcal{F}$ is a saturated fuson system on an abelian p-group P. Then $\mathcal{F} = N_{\mathcal{F}}(P)$.

The Thompson subgroup

Let p be a prime, and P a finite p-group.

The Thompson subgroup $J(P)$ of P is the subgroup of P generated by the set of abelian subgroups of P of maximal order.

$ZJ(P) := Z(J(P))$.

Theorem

(Glauberman, 71) Let p be an odd prime, let G be a finite group and let P be a Sylow p-subgroup of G. If $Qd(p)$ does not occur as a subquotient of G, then $N_G(ZJ(P))$ controls strong p-fusion in G.

$$Qd(p) := (C_p \times C_p) \rtimes SL_2(p)$$

Remark

Suppose $G = Qd(p)$, $P = (C_p \times C_p) \rtimes A$, where A is the subgroup of strict upper triangular matrices of $SL_2(p)$. Then

- $J(P) = P$
- $ZJ(P) = Z(P) = \{(x, 1) \ : x \in C_p\} \leq C_p \times C_p$
- $N_G(ZJ(P))$ *does not control fusion in G.*

Theorem

(Glauberman, '71) Let p be an odd prime, let G be a finite group and let P be a Sylow p-subgroup of G. If $Qd(p)$ does not occur as a subquotient of G, then $N_G(ZJ(P))$ controls strong p-fusion in G.

Theorem

(Linckelmann-K, '05; for blocks Linckelmann-Robinson-K, '02) Let p be an odd prime, let P be a Sylow p-subgroup of G and let $\mathcal{F}$ be a saturated fusion system on P. If $Qd(p)$ does not occur as a subquotient of the BCGLO-model $L_Q^{\mathcal{F}}$ for any $\mathcal{F}$-centric, radical, fully normalised subgroup Q of P, then $N_{\mathcal{F}}(ZJ(P)) = \mathcal{F}$.

Idea of Proof.

- Let $\mathcal{F}$ be a counter-example to the theorem with $|\mathcal{F}|$, the number of morphisms in $\mathcal{F}$ minimal.

- Use "well-placed" arguments to conclude that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some $1 \neq Q \leq P$.

- Reduce to the case that Q is $\mathcal{F}$-centric.

-By BCGLO model theorem $\mathcal{F} = N_{\mathcal{F}}(Q) = \mathcal{F}_P(L_Q^{\mathcal{F}})$.

-Apply Glauberman's theorem to the group $L_Q^{\mathcal{F}}$.

Glauberman functors

Definition

- *A positive characteristic p-functor is a map sending any finite p-group P to a characteristic subgroup $W(P)$ of P such that $W(P) \neq 1$ if $P \neq 1$ and such that any isomorphism of finite p-groups $P \cong Q$ maps $W(P)$ onto $W(Q)$.*

- *A Glauberman functor is a positive characteristic p-functor with the following additional property: whenever P is a Sylow p-subgroup of a finite group L which satisfies $C_L(O_p(L)) = Z(O_p(L))$ and such that $Qd(p)$ does not occur as a subquotient of L, then $W(P)$ is normal in L.*

- Glauberman's theorem: ZJ is a Glauberman functor.
- (Glauberman) There exist other Glauberman functors. In particular, there exist centric Glauberman functors, e.g. K_∞, K^∞.
- Can replace ZJ by any Glauberman functor.

Corollary

Let p be an odd prime, and let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If $Qd(p)$ does not occur as a subquotient of any of the BCGLO-models of $\mathcal{F}$, then $\mathcal{F}$ is non-exotic, that is $\mathcal{F} = F_P(G)$ for some finite group G.

Theorem

(Onofrei-Stancu, '08) (Generalizing a theorem of Stellmacher.) There exists a positive characteristic 2-functor $P \to W(P)$ with the following property: If $\mathcal{F}$ is a saturated fusion system on a 2-group P such that S_4 is not a subquotient of any BCGLO-model for $\mathcal{F}$, then $\mathcal{F} = N_{\mathcal{F}}(W(P))$.

Fixed point free automorphisms.

Theorem

(P.Flavell, '06) Let p be an odd prime, let G be a finite group, and P a Sylow p-subgroup of G. Suppose that G admits an automorphism α of prime order r such that r is relatively prime to $|G|$ and such that $G^{\langle\alpha\rangle}$ is a p'-group. Then $N_G(P)$ controls p-fusion in G.

Theorem

(Linckelmann-K, '07) Let p be an odd prime, $\mathcal{F}$ be a saturated fusion system on finite group P and α an automorphism of P acting freely on $P - \{1\}$, which stabilises $\mathcal{F}$ and whose order is a prime number r which does not divide the orders of the automorphism groups $\mathrm{Aut}_{\mathcal{F}}(R)$ for all $\mathcal{F}$-centric radical subgroups R of P. Then $\mathcal{F} = N_{\mathcal{F}}(P)$.

$\mathcal{F}$ is α-stable if for any $Q, R \leq P$, and any morphism $\phi : Q \to R$ in $\mathcal{F}$, the morphism $\alpha \circ \phi \circ \alpha^{-1}|_{\alpha(Q)} : \alpha(Q) \to \alpha(R)$ is a morphism in $\mathcal{F}$.

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Nilpotent fusion systems

Definition

A saturated fusion system $\mathcal{F}$ on P is nilpotent if $\mathcal{F} = \mathcal{F}_P(P)$.

Theorem

(Frobenius) Let G be a finite group, P a Sylow p-subgroup of G. The following are equivalent:

- *G is p-nilpotent, i.e. $G = O_{p'}(G) \rtimes P$.*
- *$\mathcal{F}_G(P) = \mathcal{F}_P(P)$.*

Theorem

(Linckelmann-K, '05)[Generalizing Glauberman-Thompson p-nilpotency criterion.] Let p be an odd prime and $\mathcal{F}$ a saturated fusion system on finite p-group P. Then $\mathcal{F}$ is nilpotent if and only if $N_{\mathcal{F}}(Z(J(P))$ is nilpotent.

Theorem

(Diaz-Glesser-Mazza-Park, '07) [Generalizing Thompson's p-nilpotency criterion] Let $\mathcal{F}$ be a saturated fusion system on finite p-group P. Suppose either that p is odd or that S_4 does not occur as subquotient of any BCGLO-model for $\mathcal{F}$. Then $\mathcal{F}$ is nilpotent if and only if both $N_{\mathcal{F}}(J(P))$ and $C_{\mathcal{F}}(Z(P))$ are nilpotent.

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Control of transfer

Let $\mathcal{F}$ be a saturated fusion system on finite p-group P and let $W \leq P$ be fully $\mathcal{F}$-normalized.

Definition

The $\mathcal{F}$-focal subgroup of P is the subgroup

$$[P, \mathcal{F}] = \langle u^{-1}\phi(u) : \ u \in Q, \phi \in \mathrm{Hom}_{\mathcal{F}}(\langle u \rangle, P) \rangle.$$

$N_{\mathcal{F}}(W)$ controls transfer in $\mathcal{F}$ if $[P, \mathcal{F}] = [P, N_{\mathcal{F}}(W)]$.

Theorem

(Diaz-Glesser-Mazza-Park, '08) [Generalising Glauberman's control of transfer theorem.] If $p \geq 5$, then the positive characteristic functors K^{∞} and K_{∞} control transfer in every saturated fusion system on a finite p-group P.

A consequence of Diaz-Glesser-Mazza-Park result:

Theorem

(G.Robinson, 2 April 08) Let $1 \neq P$ be a finite group such that the group of outer automorphisms, $\mathrm{Out}(P)$, of P, is a p-group. Let B be a p-block of a finite group G with P as defect group. Then the number of height 0 ordinary irreducible characters, $k_0(B)$, of B is a multiple of p.

The above theorem can be viewed as evidence toward the Alperin-McKay conjecture: Immediate from the hypothesis of $\mathrm{Out}(P)$ being a p-group that the Brauer correspondent of B is nilpotent, and hence locally the number of height 0 characters is a multiple of p.

Controlling fusion in saturated fusion system

Radha Kessar

(Slide presentation)

Slide 7/30 :

Extension Axiom:

$\mathcal{F} = \mathcal{F}_p(G) \qquad Q, R \leq P$

$\varphi : Q \xrightarrow{\cong} R$ induced by $g \in G$

$y \longmapsto gyg^{-1}$

$g N_\varphi g^{-1} = N_p(R) C_G(R) \cap g N_p(Q) g^{-1}$

a p subgp of $N_p(R) C_G(R)$.

· $N_p(R)$ is a Sylow p-subgp of $N_p(R) C_G(R)$.

$\exists \; x \in N_p(R) C_G(R)$ s.t $x g N_\varphi g^{-1} x^{-1} \leq N_p(R)$

Can assume $x \in C_G(R)$.

Let $\hat{\varphi} : N_\varphi \longrightarrow P$

$y \longmapsto xg y g^{-1} x^{-1}$