Riemannian Submersions and Foliations

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§0. Riemannian Submersions (see Besse, Ch 9 + O'Neill [1968])

<u>Def'n</u>: $\pi : (Q^N, g) \to (M^m, \bar{g})$ is <u>R.s.</u> if $\pi'(q) : T_q Q \to T_{\pi(q)} M$ is surj. <u>orth. proj.</u> for all $q \in Q$. Also define Riemannian foliation?

<u>Many, many examples</u>: (a) (Warped) Products
(b) Given $\pi : Q^N \to M^m$ and $\bar{g}$ on M, there are many, many g on Q s.t. π is R.S.
(c) Compositions of R.s. are R.s.

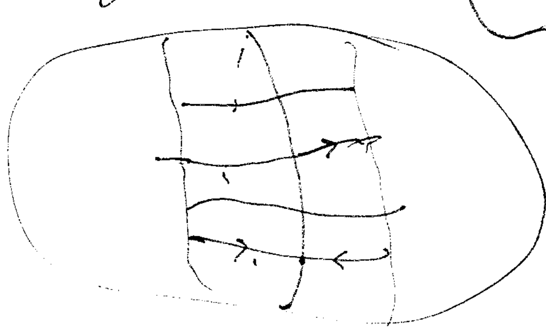
<u>Interesting problem</u>: For a fixed (Q^N, g), classify the R.s. $\pi : (Q^N, g) \to (M^m, \bar{g})$.

Problem has local and global aspects:

<u>Easy local aspect</u>: $m = 1$. Can take $M = \mathbb{R}^1$ or S^1 with standard metric and we are reduced to solving the eikonal eqn

$$\|d\pi\|_g = 1. \qquad \pi : (Q, g) \to \mathbb{R}. \quad (\text{or } S^1)$$

Local soln is trivial: Fix an (non-oriented) hypersurface $H \subseteq Q$ and let $U \subseteq Q$ be a domain containing H with ~~the~~ a retraction along normal geodesic map



Then $\pi(q) =$ signed $\operatorname{dist}(q, H)$ (signed (oriented))

Globally you need H w.o. focal pts.

$\boxed{\text{From now on } m > 1,}$ $\boxed{\text{set } n = N - m > 0}$

Examples still easy to come by:

If ~~Lie gp~~ H acts on Q w isomog m, then

$M = H\backslash Q$ will be a mfd "most places", when Hausdorff.

and will ~~...~~ the proj $\pi : Q \to M$ will be R.s. w.r.t. a single metric $\bar{g}$ on M.

So "classification" in the naive sense would included cohomog m group actions.
as well as fiber group related $K \subseteq H \subseteq G$ $G/K \to G/H$

<u>This is by no means all the possibilities though</u>. Even for $N=3, m=2$, it is not so easy to say when (Q^3, g) has a rank 2 R.s., even locally <u>or</u> how many there are.

<u>Local</u>: If $\pi : (Q^3, g) \to (M^2, \bar{g})$ is R.s. can always choose local coords (x^1, x^2, x^3) centred on $q \in Q$ s.t.

$$\pi(x^1, x^2, x^3) = (x^2, x^3)$$

and

$$g = (dx^1 + f_2(x^1,x^2,x^3)\,dx^2 + f_3(x^1,x^2,x^3)\,dx^3)^2 + E(x^2,x^3)(dx^2)^2 + 2F(x^2,x^3)\,dx^2 dx^3 + G(x^2,x^3)(dx^3)^2.$$

so, mod diffeos, ~~the~~ R.s. with $(N, m) = (3, 2)$ depend on 2 fctns of 3 vars.
← also can think of 3 1st order eqns for unit v.f. still

| 3 eqns for the 3 functions of (x^2, x^3) open dist. |

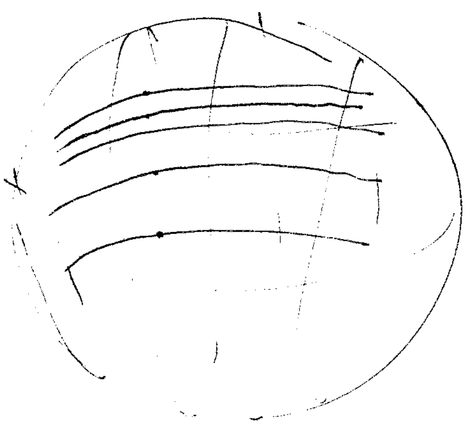
How to recognize that there are such coords by conditions on g? Don't know all the conditions. At least 4 derivates of g are needed, ~~and probably more~~ before you see any condition.

Let (Q_0^3, g) have constant sectional curvature K. Two kinds of examples:

E(i) Let $H \subseteq \mathrm{Iso}\,(Q,g)$ be a 1-parameter subgroup and set

$$M^2 = H \backslash Q$$

E(ii) Let $U \subseteq Q^3$ be foliated by 2-planes (= tot. geod. surfaces) $\mathcal{M}$ on U



Let $\mathcal{F}$ be the orthogonal foliation by curves.

Then the quotient $M^2 = Q^3/\mathcal{F}$. is a surface and the map $\pi : U \to M$ is a (local) R.S..

$G_2(Q^3)$ = ge tot. geod surfaces in Q^3 (a 3-mfd).

<u>Proposition</u>: Let (Q^3, g) have constant sect'l curvature K and let $\pi : Q \to M$ be smooth R.S. ~~then~~ Let $\mathcal{H} = (\ker \pi')^{\perp}$ be the horizontal m plane field

(i) If $\mathcal{H}$ is contact then π is locally like E(i)

(ii) If $\mathcal{H}$ is integrable, then π is locally like E(ii).

The proof is fairly easy. The harder part is case 1, actually. You must differentiate several times to prove that the fibers are the orbits of a Killing field.

$$
\begin{tikzcd}
 & Z \arrow[dl] \arrow[dr, "\rho"] & \\
G_m(Q^{n+m}) & & Q^{n+m} \\
\cup & \lambda^{-1}(\Lambda) \supseteq \tilde{U}^{n+m} \arrow[dl] \arrow[dr, "\wr\,\rho"] & \cup \\
\Lambda^n & & U^{n+m} \arrow[ll, "\lambda"'] \arrow[d, "\pi"] \\
 & & M^m
\end{tikzcd}
$$

$\bar{g}$ has sec. curve $\equiv K$.

<u>Theorem</u>: When $n = 2$ $\quad$ I is involutive with $s_2 = 1$ the ~~last~~ highest non-zero Cartan character

When $n > 2$ $\quad$ $I^{(1)}$ is involutive ~~of~~ on $V_n^0(G_m(Q^{n+m}), I)$ with ~~last~~ highest nonzero Cartan character $s_2 = \binom{n}{2}$.

<u>Cor.</u> ~~So~~ When $n > 1$, the "generic" hm. int. R.s. $\pi: U \to M^m$ depends on $\binom{n}{2}$ functions of 2 variables.

Can say much more, but I won't at the moment....

From now on assume $p \neq 0$ on U.

For all n. **Prop**: If $n = 1$ then π is locally 1-parameter group quotient.

What about first nontrivial case? $N = 4$ $(n, m) = (2, 2)$

Prop: Let $\pi : U^4 \to M^2$ be a R.s. and suppose that ~~[?]~~ $\mathcal{H}' = \mathcal{H} \oplus [\mathcal{H}, \mathcal{H}]$ has rank 3. Then $\mathcal{H}'$ is integrable, there exist a R.s. $\pi' : U^4 \to \hat{U}^3$ with horizontal plane field $\mathcal{H}'$ and a R.s. $h : \hat{U}^3 \to M^2$ that is a 1-parameter group quotient s.t. $\pi = h \circ \pi'$.

$\boxed{\text{Define prime}}$

$N = 5$.

Prop: Let $\pi : U^5 \to M^3$ be a ~~R.s.~~ smooth R.s. Then locally outside a closed set with no interior, $\pi = h \circ \pi'$ where $\pi' : U^5 \to \hat{M}^{3+k}$ is a R.s. with ~~has~~ $\mathcal{H}'$ integrable and $h : \hat{M}^{3+k} \to M^3$ is ~~reduction by~~ a reduction by a cohomog 3 ~~inf. gp~~ local group action on $\hat{U}^{3+k}$.

Remark: Each of the cases $k = 0, 1, 2$ occurs.

For the $N = 5$ $(n, m) = (3, 2)$

$$\mathcal{H}' = \overset{\scriptscriptstyle 4\,5}{\mathcal{H}} \oplus \overset{\scriptscriptstyle 3}{[\mathcal{H}, \mathcal{H}]} = \mathcal{H} \oplus L \quad \text{orthog sum:}$$

$$\mathcal{V} = L \oplus \mathcal{V}'$$

Let $A : \underset{2}{\mathcal{H}} \to \underset{2}{\mathcal{V}'}$ be given by $A(X) = [E, X]^{\mathcal{V}'}$ $|E| = 1$.

A is linear trans has singular values $0 \leq \sigma_1 \leq \sigma_2$.

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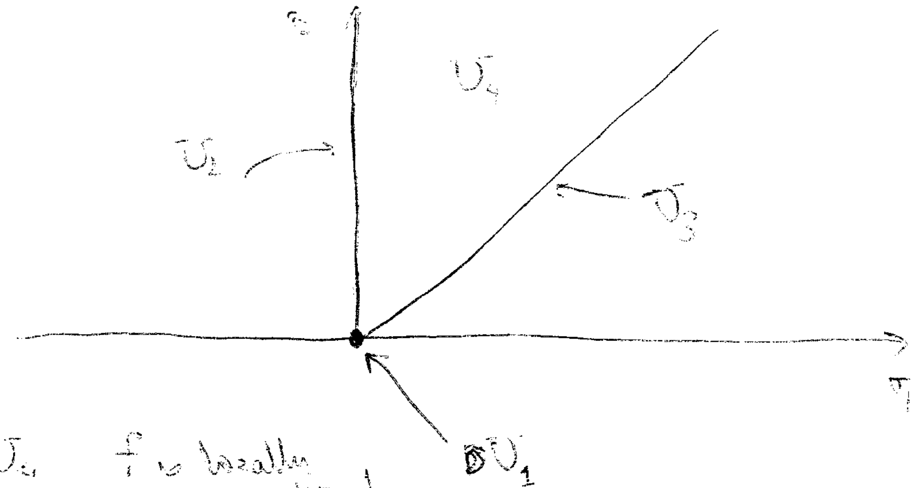
$U = S \cup U_1 \cup U_2 \cup U_3 \cup U_4$



on U_4 f is locally [illegible]

$U_2 = U_3 = \emptyset$

on U_1 either (i) U' is integrable and things factor

$U_1 \longrightarrow$ [illegible] $\quad U_1 \xrightarrow{\bar{\ }} \bar{U}_1' \subseteq \mathbb{Q}^2_K$

$\pi \searrow \quad \downarrow h \leftarrow$ [illegible]

M^2

(ii) U' is not integrable and π is [illegible]

Moreover, these depend on [illegible] of [illegible]

$(a + y^2) = (b + z^2) f($