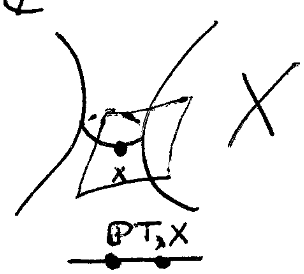
Notes by M.S.

J. M. Landsberg – Lines and Asymptotic Lines of Projective Varieties

I) Asymptotic Lines
  A) Definition
  B) Projective rigidity revisited
II) Motivation: The Hwang-Mok Program
  A) Fano variations via their rational class
  B) Successes / Hopes
  C) Hwang's Question
III) Our answers
  A) How $\mathcal{C}_{k,x}$ moves
  B) How $\mathcal{C}_x$ moves
IV) Future work
  A) Theoretical Computer science $P \overset{?}{=} NP$
  B) Algebraic geometry: Debarre-de Jong Conj.
  C) Extend techniques to intrinsic settings.

$V = \mathbb{C}^{n+a+1}$



$X^n \subset \mathbb{P}V$

$x \in X$

1st Derivatives $\mathbb{P}T_xX \subset \mathbb{P}T_x\mathbb{P}V$
  ~ tangent directions to lines ($\mathbb{P}^1$'s) having contact to X to order ≥ 1 at x

2nd Derivatives "asymptotic directions" $\mathcal{C}_{2,x} \subset \mathbb{P}T_xX$
  ~ tangent dir. ... order ≥ 2

etc... $\mathcal{C}_{k,x} \subset \cdots \subset \mathcal{C}_{2,x} \subset \mathbb{P}T_xX$

$\mathcal{C}_{\infty,x} = \mathcal{C}_x :=$ tangent directions to lines on X thru $x \in X$

Colleen Robles

B) Projective rigidity (C.R.'s talk)

ex We say $Z = G/P \subset \mathbb{P}V$ is rigid at order two if for any $X^n \subset \mathbb{P}V$ w/ $\mathcal{C}_{2,x,X} \cong \mathcal{C}_{2,\xi,X}$ $x \in X_{\text{rigid}}$

$\Longrightarrow X \cong Z$

ex $Z = G(k,W) \subset \mathbb{P}(\Lambda^k W)$

$T_E G(k,W) \simeq E^* \otimes W/E$

$\mathcal{C}_{E,Z} = \mathcal{C}_{2,E,Z} \simeq \mathrm{Seg}(\mathbb{P}E^* \times \mathbb{P}(W/E))$

ex $Z = Q^n \subset \mathbb{P}^{n+1}$

$\mathcal{C}_{2,\xi,Z} \simeq Q^{n-2} \subset \mathbb{P}T_\xi Z = \mathbb{P}^{n-1}$

But any smooth hypersurface has $\mathcal{C}_{2,x} \simeq Q^{n-2}$

ex $Z^3 = \mathrm{Flags}_{1,2}(\mathbb{C}^3) \subset \mathbb{P}(sl_3)$

$\mathcal{C}_{2,x} \subset H \subset \mathbb{P}T_3 Z$

$\mathcal{C}_{2,x}$ = 2 pts., $H \simeq \mathbb{P}^1$, $\mathbb{P}T_3 Z = \mathbb{P}^2$

2 types of imposters
- intrinsically flat
- not intrinsically flat

II) Hwang-Mok

$L \to Z$ compact complex manifold

$\forall \xi \in Z \ \exists f : \mathbb{P}^1 \to Z, \ f(0) = \xi$

Let $\tilde{\mathcal{C}}_\xi \subset \mathbb{P}T_\xi Z$ denote tangent dirs. to minimal rat. curves on Z thru ξ

$Z = X \subset \mathbb{P}V$ uniruled by lines

$\tilde{\mathcal{C}}_\xi = \mathcal{C}_\xi$

<u>H-M Program</u> Study Z via $\tilde{\mathcal{C}}_\xi$

$$\begin{matrix} U & & U' \\ \cap & & \cap \\ Z & & Z' \end{matrix}$$

HWang's question:

$X^n \subset \mathbb{P}^{n+1}$ $n \geq 4$ smooth cubic hypersurface

$x \in X_{gen'l}$ $\mathcal{C}_x \subset \mathbb{P}^{n-1}$ codim 2, deg 6

How can $\mathcal{C}_x$ vary as one moves $x \in X$?

$T = \mathbb{C}^n$ fix $Q \in S^2T^*$

$$\mathcal{C} \in \mathcal{M}_3 = \left\{ \begin{matrix} \text{space of codim 2} \\ \text{deg 6} \subset \mathbb{P}T \end{matrix} \right\} = \left. \left(S^3T^* / Q \circ T^* \right) \right/ SO(T,Q)$$

$\mathcal{C} = \{Q = 0, P_3 = 0\}$

$X \dashrightarrow \mathcal{M}_3$

More general Question: $X^n \subset \mathbb{P}^{n+1}$ hypersurface degree d

have $\varphi_k : X \dashrightarrow \mathcal{M}_k$ = complete intersections of deg $2, \dots, k$ in $\mathbb{P}T$

$$\begin{tikzcd} \mathcal{F} \subset GL(V) \arrow[r, dashed, "\tilde{\varphi}_k"] \arrow[d, "\text{adapted frame bundle of } x"'] & S_k \subset \bigoplus_{\delta=3}^{k} S^\delta T^* \arrow[d] \\ X \arrow[r, dashed, "\varphi_k"'] & \mathcal{M}_k \end{tikzcd}$$

open subset

$$\begin{matrix} P = (Q, P_3, \dots, P_k) \\ \downarrow \\ \mathcal{C}_{[P]} \end{matrix}$$

$$T_P S_k = \bigoplus_{\delta=3}^{k} S^\delta T^*$$

1^{st} case where assume $d\varphi_x$ injective
Need to study $\underline{G}(n, T\mathcal{M}_k)$

Write $\tilde{R} = \langle R_1, \dots, R_n \rangle$ $\quad R_j \in \mathcal{S}_k$

$$[\tilde{R}] \in G(n, T_p\mathcal{M}_d) \qquad R_j = (R_{j3}, \dots, R_{jk})$$

$\underline{Q)}$ $\exists$? restrictions on $[\tilde{R}]$ to have $[\tilde{R}] = d\varphi_x(T_xX)$ $\quad x \in X_{gen'l}$?

Answer Yes

$\underline{\text{Thm 1}}$ (Rough version)
* All terms of deg $< k$ in $[\tilde{R}]$ are determined by P
* $\exists\ M_{k+1} \in S^{k+1}T^*$ s.t. $\langle R_{1,k}, \dots, R_{n,k} \rangle = \{\text{derivative of } M_{k+1}\}$

i.e. Let $\tilde{\Sigma} \subset \underline{G}(n, T\mathcal{M}_k)$ then $\gamma_{\varphi_k}(X) \subset \tilde{\Sigma}$

i.e. $\varphi(X)$ is an integral manifold of EDS on $\mathcal{M}_k$ defined by restricting tault system on $\underline{G}(n, T\mathcal{M}_k)$ to $\tilde{\Sigma}_k$

$$\gamma_{\varphi_k} : X \dashrightarrow \underline{G}(n, T\mathcal{M}_d)$$

$$\xi_\delta : \mathcal{S}_k \to S^{\delta+1}T^*$$
$$P \longmapsto \mathrm{Sym}(Q^\# \lrcorner (P_\varepsilon \otimes P_3))$$

$\underline{\text{Thm 1}}$ need $R_{\delta+1} = \xi_\delta(P) + P_{\delta+1}$ $\qquad \delta+1 < k$

Do we get additional conditions when $k = d = \deg(X)$?

$\underline{\text{Thm 2}}$ Yes

Thm 2 get additional condition

$$\mu_{d+1} \in I_{d+1}(\mathcal{C}_{[P]}) \subset S^{d+1}T^*$$

derivs of

$$\underline{\underline{\{\overset{\epsilon}{S}_d(P) + \mu_{d+1}\}}}$$

Thm 3 $X^n \subset \mathbb{P}^{n+1}$ $\quad x \in X_{gen'l}$ $\quad \Phi \subset \boxed{\sim}$

assume $\gamma_{\varphi_d}(x) \in \Phi$ $\quad \gamma^c$ smooth

$\Longrightarrow \deg(X) = d$

$\underline{\mathcal{M}_3}$

$n=4$ space of genus 4 curves in canonical embedding

$n=5$ K3 surfaces

Thm 4 $\exists$ Zar open subset $\mathcal{U}_d \subset \mathcal{M}_d$ s.t.

if $x \in X_{gen'l}$ and $\varphi(x) \in \mathcal{U}_d$

$\Longrightarrow \operatorname{rank} d\varphi_p = n$ $\quad \nabla$

$$\mathcal{F}^2 \subset GL(V)$$
$$\downarrow \qquad \omega = (\omega^A_B)$$
$$X \subset \mathbb{P}V$$

$$F_k = r_{\alpha_1 \dots \alpha_k}\, \omega^{\alpha_1} \circ \dots \circ \omega^{\alpha_k} \in P(\mathcal{F}^2, \pi^*(S^kT^*X))$$

$$0 = dq_{\alpha\beta} = r_{\alpha\beta\gamma}\,\omega^\gamma + (\omega^\alpha_\beta + \omega^\beta_\alpha) + \text{stuff}$$

$$dr_{\alpha\beta\gamma\delta}\,\omega^\delta + 3 r_{\epsilon(\alpha\beta}\,\omega^\epsilon_{\gamma)} + \text{stuff}$$

$$\omega^\alpha_\beta = \tfrac{1}{2}(\omega^\alpha_\beta + \omega^\beta_\alpha) + \tfrac{1}{2}(\omega^\alpha_\beta - \omega^\beta_\alpha)$$

$$dr_{\alpha_1 \dots \alpha_p} = \Big(r_{\alpha_1 \dots \alpha_p \beta} + \frac{p}{2} \sum_\gamma r_{\gamma\gamma(\alpha_1 \dots \alpha_{p-1}} r_{\gamma\beta)}\Big)\,\omega^\beta_\alpha + \text{stuff}$$

IV A) $\det_n \in S^n\mathbb{C}^{n^2}$ $\quad$ $\mathrm{perm}_m \in S^m\mathbb{C}^{m^2}$
$\in VP$ $\quad$ $\cap VNP$

Find $n(m)$ s.t. $\{\mathrm{perm}_m = 0\}$ is an affine linear section of $\{\det_{n(m)} = 0$

Known $\exp(m) \geq n(m) \gtrsim m^2/2$