Infinitesimal isospectral deformations of symmetric spaces

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We are interested in studying the infinitesimal isospectral deformations of the Riemannian symmetric spaces of compact type and determining which spaces are isospectrally rigid to first-order.

Let (X, g) be a compact Riemannian manifold. The spectrum $\mathrm{Spec}(X, g)$ of (X, g) is the set of the eigenvalues of the Laplacian counted with their multiplicities. Let $\{g_t\}$ be family of Riemannian metrics on X, with $g_0 = g$. We say that the family $\{g_t\}$ is an isospectral deformation of g if the spectrum $\mathrm{Spec}(X, g_t)$ of the Laplacian of the metric g_t is independent of the parameter t. The family $\{g_t\}$ is trivial if there exists a family of diffeomorphisms φ_t of X such that $g_t = \varphi_t^* g$, for all t; in this case, all the manifolds (X, g_t) has the same spectrum. The infinitesimal deformation of $\{g_t\}$ is the symmetric 2-form

$$h = \frac{d}{dt} g_{t|t=0};$$

we say that h is trivial if it of the form $\frac{d}{dt}\varphi_t^* g_{|t=0}$, where $\{\varphi_t\}$ is one-parameter family of diffeomorphisms of X, or equivalently if it is a Lie derivative of the metric g.

We henceforth suppose that (X, g) is a symmetric space of compact type. Guillemin showed that the infinitesimal deformation of an isospectral deformation of g satisfies a certain integral condition, namely, that it belongs to the kernel of a certain Radon transform.

We say that a symmetric 2-form h on X satisfies the Guillemin condition if, for every maximal flat totally geodesic torus Z contained in X and for all parallel vector fields ζ on Z, the integral

$$\int_Z h(\zeta, \zeta)\, dZ$$

vanishes, where dZ is the Riemannian measure of Z.

Guillemin's result may stated as:

THEOREM 1. *The infinitesimal deformation of an isospectral deformation of a symmetric space of compact type satisfies the Guillemin condition.*

A symmetric 2-form, which is a Lie derivative of the metric, always satisfies the Guillemin condition.

We consider the space $\mathcal{N}$ of all symmetric 2-forms satisfying the Guillemin condition and define the space of infinitesimal isospectral deformations of X to be the orthogonal complement of the space $\mathcal{L}$ of Lie derivatives of the metric g in $\mathcal{N}$; we have the orthogonal decomposition

$$\mathcal{N} = \mathcal{L} \oplus I(X).$$

If the space $I(X)$ vanishes, we say that the space (X, g) is infinitesimally rigid in the sense of Guillemin; under this assumption, an isospectral deformation of the metric g is trivial to first-order and the space X is infinitesimally spectrally rigid (i.e., spectrally rigid to first-order).

We consider the tangent bundle T, the cotangent bundle T^* and the bundle S^2T^* of all symmetric 2-forms on X. The Killing operator

$$D_0 : C^\infty(T) \to C^\infty(S^2T^*)$$

is the first-order differential operator sending a vector field ξ into the Lie derivative of g along ξ. The Guillemin rigidity of X is equivalent to the following statement: the only symmetric 2-forms on X satisfying the Guillemin condition are the Lie derivatives of the metric g, or equivalently, the equality

$$D_0(C^\infty(T)) = \mathcal{N}$$

holds.

1. The spaces of rank one

All the previously known spectral rigidity results for symmetric spaces with positive curvature concern spaces of rank one, i.e., the spheres and projective spaces.

For $2 \leq n \leq 6$, the spectral rigidity of the sphere S^n was established by Berger and Tanno. Also Tanno proved that a metric g' on S^n, with $n \geq 2$, which is sufficiently close to g and satisfies $\mathrm{Spec}(X, g') = \mathrm{Spec}(X, g)$, is isometric to g.

Let (X, g) be a space of rank one endowed with its canonical metric; the metric g is a C_l-metric: all of its geodesics are closed and have the same length l.

A result of Duistermaat–Guillemin implies the following:

THEOREM 2. *A metric on a space* (X, g) *of rank one whose spectrum is equal to the spectrum of the metric* g *is a* C_l*-metric.*

Since the flat tori of X are its closed geodesics, an elementary computation shows that the infinitesimal deformation of g by C_l-metrics satisfies the Guillemin condition. Thus in this case, Theorem 2 implies Theorem 1. The Guillemin condition was first introduced by Michel for these spaces in conjunction with the Blaschke conjecture.

From Theorem 2 and Green and Berger's positive resolution of the Blaschke conjecture for the real projective spaces, we obtain the spectrally rigidity of the real projective space $\mathbb{RP}^n$.

The sphere S^n is not rigid in the sense of Guillemin, because all odd forms on S^n satisfy the Guillemin condition.

The symmetric spaces of rank one which are not spheres are rigid in the sense of Guillemin [R. Michel (1973) for the real projective spaces $\mathbb{RP}^n$; Michel and Tsukamoto (1981) for the other projective spaces].

A spectral rigidity result for a projective space X which is a not sphere can be derived from its Guillemin rigidity by means of Kiyohara's work on C_π-metrics and Theorem 2: a metric g' on X, whose spectrum is equal to that of g and which is sufficiently close to g, is isometric to g.

2. Necessary conditions for infinitesimal rigidity

THEOREM 3. *A product of irreducible symmetric spaces is not rigid in the sense of Guillemin.*

We henceforth suppose that X is irreducible. The reduced space of X plays a crucial role; this symmetric space is covered by X and is not the cover of another symmetric space. We say that X is reduced if it is equal to its reduced space. The reduced space of the n-sphere S^n is the real projective space $\mathbb{RP}^n$. The reduced space of a compact simple Lie group G is the quotient G/Z of G by its center Z.

We find the following necessary condition for rigidity in the sense of Guillemin:

THEOREM 4. *If an irreducible symmetric space X is infinitesimally rigid in the sense of Guillemin, then X is reduced.*

In fact, if X is an irreducible space which is not reduced, then X always possesses an isometry which give rise to symmetric 2-forms which lie in the kernel $\mathcal{N}$ of our Radon transform and which are not Lie derivatives of the metric. For the sphere S^n, the maximal flat tori are the closed geodesics (i.e. the great circles) and the odd symmetric 2-forms all belong to $\mathcal{N}$.

Thus the relevant problems concerning infinitesimal isospectral deformations for our class of symmetric spaces may be formulated as follows: *determine which irreducible reduced spaces are rigid in the sense of Guillemin; more generally, determine the space of infinitesimal isospectral deformations of an irreducible reduced space.*

3. The Grassmannians

Let $\mathbb{K}$ be a division algebra over $\mathbb{R}$ (i.e. $\mathbb{K}$ is equal to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) and let $m, n \geq 1$ be given integers. The Grassmannian $G^{\mathbb{K}}_{m,n}$ of all $\mathbb{K}$-planes of dimension m in $\mathbb{K}^{m+n}$ is an irreducible symmetric space of compact type whose rank is $\min(m,n)$, with the exception of $G^{\mathbb{R}}_{1,1} = S^1$ and $G^{\mathbb{R}}_{2,2}$; the universal covering space of $G^{\mathbb{R}}_{2,2}$ is $S^2 \times S^2$.

Our first result concerning spaces of higher rank is the following:

THEOREM 5. *For $m, n \geq 2$, with $m \neq n$, the Grassmannian $G^{\mathbb{K}}_{m,n}$ is rigid in the sense of Guillemin.*

This result implies that the Grassmannians of Theorem 5 are infinitesimally spectrally rigid and provides us with the first examples of irreducible symmetric spaces of arbitrary rank having this property.

This theorem together with Theorem 4 and the results about spaces of rank one tell us that:

An irreducible symmetric space, which is a Grassmannian, is rigid if and only if it is reduced.

The Grassmannian $G^{\mathbb{K}}_{n,n}$, with $n \geq 2$, is not reduced; it is a two-fold covering of its reduced space $\bar{G}^{\mathbb{K}}_{n,n}$.

Since the reduced space $\bar{G}^{\mathbb{C}}_{2,2}$ is isometric to the real Grassmannian $G^{\mathbb{R}}_{2,4}$, it is rigid in the sense of Guillemin.

4. Infinitesimal isospectral deformations

THEOREM 6. *The reduced space $\bar{G}^{\mathbb{R}}_{3,3}$ of the Grassmannian $G^{\mathbb{R}}_{3,3}$ of 3-planes in $\mathbb{R}^6$ is not rigid in the sense of Guillemin.*

The space $\bar{G}^{\mathbb{R}}_{3,3}$ is the first example of an irreducible reduced symmetric space which is not rigid in the sense of Guillemin.

As we shall show, the non-rigidity of this space arises from an invariant symmetric 3-form and leads us to ask the following question:

Which irreducible symmetric spaces admit non-zero symmetric 3-form invariant under its group of isometries? The answer is given by the following result:

THEOREM 7. *The only irreducible simply-connected symmetric spaces which admit a non-zero symmetric 3-form invariant under its group of isometries are given by the following list:*

(i) $SU(n)$, *with* $n \geq 3$;

(ii) $SU(n)/SO(n)$, *with* $n \geq 3$;

(iii) $SU(2n)/Sp(n)$, *with* $n \geq 3$;

(iv) E_6/F_4.

Here we shall consider only the first three families of spaces. Let $\tilde{X}$ be one of the spaces $SU(n)$, $SU(n)/SO(n)$ or $SU(2n)/Sp(n)$, with $n \geq 3$. Its reduced space X is a quotient of $\tilde{X}$ by a cyclic group of isometries of order n.

If $\tilde{X} = SU(n)$, then X is the group $SU(n)/S$, where S is the center of $SU(n)$. The space $SU(n)/SO(n)$ is the special Lagrangian Grassmannian and we call its quotient Y_n the reduced Lagrangian Grassmannian. We denote by Z_n the reduced space of $SU(2n)/Sp(n)$.

THEOREM 7. *The reduced spaces $SU(n)/S$, Y_n and Z_n, with $n \geq 3$, are not rigid in the sense of Guillemin.*

The special Lagrangian Grassmannian $SU(4)/SO(4)$ is isometric to the universal cover of the Grassmannian $G^{\mathbb{R}}_{3,3}$, and hence Y_4 is isometric to $\bar{G}^{\mathbb{R}}_{3,3}$.

We write $G = SU(n)$ and consider the real-valued G-invariant polynomial Q_p on the Lie algebra $\mathfrak{su}(n)$ of G defined by

$$Q_p(A) = (-i)^p \operatorname{Tr} A^p,$$

for all $A \in \mathfrak{su}(n)$. Let X be one of the above reduced symmetric spaces. It admits a non-zero G-invariant symmetric p-form σ_p arising from the G-invariant polynomial Q_p; in fact, σ_2 is the metric of X and the 3-form $\sigma = \sigma_3$ is up to a constant the only G-invariant symmetric 3-form on X. Since X is irreducible, the form σ induces an injective morphism

$$\tilde{\sigma} : T^* \to S^2 T^*.$$

We say that a 1-form θ on X satisfies the Guillemin condition if, for every maximal flat totally geodesic torus Z contained in X and for all parallel vector fields ζ on Z, the integral

$$\int_Z \theta(\zeta)\, dZ$$

vanishes, where dZ is the Riemannian measure of Z.

PROPOSITION 8. *A 1-form θ on X satisfies the Guillemin condition if and only if the 2-form $\tilde{\sigma}(\theta)$ satisfies the Guillemin condition.*

Therefore if $C^\infty(X)$ is the space of all real-valued functions on X and P is the orthogonal projection of $\mathcal{N}$ onto $I(X)$, the mapping

$$P\tilde{\sigma}d : C^\infty(X) \to I(X)$$

is well-defined. We construct an explicit subspace $\mathcal{F}$ of $C^\infty_{\mathbb{R}}(X)$ of finite codimension.

THEOREM 9. *Let X be the group $SU(n)/S$, or the reduced Lagrangian Grassmannian Y_n, or the symmetric space Z_n, with $n \geq 3$. The restriction*

$$P\tilde{\sigma}d : \mathcal{F} \to I(X)$$

is injective, and the space X is not rigid in the sense of Guillemin.

THEOREM 10. *Let X be the group $SU(3)/S$, or the reduced Lagrangian Grassmannian Y_3. Then the mapping*

$$P\tilde{\sigma}d : \mathcal{F} \to I(X) \tag{1}$$

is an isomorphism.

These results lead us to make the following conjecture:

Conjecture. (i) *An irreducible reduced symmetric space is rigid in the sense of Guillemin if and only if it does not admit an invariant symmetric* 3*-form.*

(ii) *If X is an irreducible reduced symmetric space which admits an invariant symmetric* 3*-form σ, the mapping* (1) *is an isomorphism.*

Let E be the trivial real line bundle over X; we consider the differential operator

$$D_\sigma : C^\infty(T) \oplus C^\infty(E) \to C^\infty(S^2T^*)$$

defined by

$$D_\sigma(\xi, f) = D_0\xi + \tilde{\sigma} df,$$

for $\xi \in C^\infty(T)$ and $f \in C^\infty(E)$. According to Proposition 8, the image of D_σ is a subspace of $\mathcal{N}$. In view of Theorem 9, to prove that the mapping (1) is an isomorphism, it suffices to show that

$$D_\sigma\big(C^\infty(T) \oplus C^\infty(E)\big) = \mathcal{N}.$$

5. The maximal flat Radon transform

Let $X = G/K$ be an irreducible symmetric space of compact type, where G is a compact connected semi-simple Lie group, which acts on X by isometries, and K is the isotropy group of a point of X. Let $C^\infty_{\mathbb{R}}(X)$ be the G-module consisting of all real-valued functions on X.

The space Ξ of all maximal flat totally geodesic tori of X is a homogeneous space of G.

The maximal flat Radon transform of X is a G-equivariant linear mapping:

$$I : C^\infty_{\mathbb{R}}(X) \to C^\infty(\Xi);$$

it assigns to a function f on X the function $\hat{f}$ on Ξ, whose value at a torus $Z \in \Xi$ is the integral

$$\hat{f}(Z) = \int_Z f\, dZ.$$

The space $\mathcal{N}$ of all symmetric 2-forms on X satisfying the Guillemin condition is a G-submodule of $C^\infty(S^2T^*)$ and may be viewed as the kernel of a maximal flat Radon transform, which is a G-equivariant linear mapping

$$I_2 : C^\infty(S^2T^*) \to C^\infty(\Xi, L_2),$$

where L_2 is a certain homogeneous vector bundle over Ξ.

PROPOSITION 11. *Let X be an irreducible symmetric space of compact type.*

(i) *If X is rigid in the sense of Guillemin, the maximal flat Radon transform (for functions) on X is injective.*

(ii) *If the maximal flat Radon transform (for functions) on X is injective, then the space X is reduced.*

EXAMPLE: The kernel of the maximal flat Radon transform for functions on the sphere S^n is the space of all odd functions. This Radon transform is injective when restricted to the even functions on S^n; this is equivalent to the classic fact that the Radon transform for functions on its reduced space $\mathbb{RP}^n$ of S^n is injective.

Grinberg conjectured the following result which generalizes the results concerning S^n and $\mathbb{RP}^n$:

If the irreducible symmetric space X is reduced, the maximal flat Radon transform for functions on X is injective.

We have verified this result for all the symmetric spaces which we have considered and for the Grassmannians $\bar{G}^{\mathbb{K}}_{n,n}$, other than $\bar{G}^{\mathbb{R}}_{2,2}$.

6. Harmonic analysis

We suppose that the space X is reduced. We consider the isotypic component $C^\infty_\gamma(F)$ of a complex homogeneous vector bundle F over the homogeneous space X corresponding to an element γ of the dual $\hat{G}$ of the group G. We denote by $\operatorname{Mult} C^\infty_\gamma(F)$ the multiplicity of the G-module $C^\infty_\gamma(F)$. We denote by $E_{\mathbb{C}}$ the complexification of a real bundle E over X. If F is the trivial complex line bundle, we identify $C^\infty(F)$ with the space $C^\infty(X)$ of complex-valued functions on X and we write $C^\infty_\gamma(X) = C^\infty_\gamma(F)$.

PROPOSITION 12. *The maximal flat Radon transform on the space (X, g) is injective if and only if the restriction of I to $C^\infty_\gamma(X)$ is injective for all $\gamma \in \hat{G}$.*

For $\gamma \in \hat{G}$, the space $C^\infty_\gamma(X)$ either vanishes or is an irreducible G-module. Since $\bigoplus_{\gamma\in\Gamma} C^\infty_\gamma(X)$ is a dense subspace of $C^\infty(X)$, in order to prove the injectivity of the maximal flat Radon transform I on X, it suffices to show that to carry out the following:

Whenever the G-module $C^\infty_\gamma(X)$, with $\gamma \in \Gamma$, is non-zero, construct an explicit non-zero vector f_γ of this G-module and prove that the function $\hat{f}_\gamma$ is non-zero.

The Killing operator

$$D_0 : C^\infty(T) \to C^\infty(S^2T^*)$$

is homogeneous, and so

$$D_0 C^\infty_\gamma(T_{\mathbb{C}}) \subset C^\infty_\gamma(S^2T^*_{\mathbb{C}}).$$

We consider the complexification $\mathcal{N}_{\mathbb{C}} \subset C^\infty(S^2T^*_{\mathbb{C}})$ of the space $\mathcal{N}$; it consists of all complex symmetric 2-forms on X satisfying the Guillemin condition. From the facts that $\bigoplus_{\gamma\in\Gamma} C^\infty_\gamma(S^2T^*_{\mathbb{C}})$ is a dense subspace of $C^\infty(S^2T^*_{\mathbb{C}})$ and that D_0 is an elliptic differential operator, we infer that

Proposition 13. *The space (X, g) is rigid in the sense of Guillemin if and only if, for all $\gamma \in \hat{G}$, we have the equality*

$$\mathcal{N}_{\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_{\mathbb{C}}) = D_0 C^\infty_\gamma(T_{\mathbb{C}}).$$

We now suppose that X is not equal to a simple Lie group. Then the complexification $\mathfrak{g}$ of the Lie algebra of G is simple. Let γ_1 be the element of $\hat{G}$ which corresponds to the irreducible G-module $\mathfrak{g}$.

To prove the Guillemin rigidity of X, it is sufficient to:

(i) For all elements $\gamma \in \hat{G}$, determine the multiplicities of the G-modules $C^\infty_\gamma(T_\mathbb{C})$ and $C^\infty_\gamma(S^2T^*_\mathbb{C})$.

(ii) For all $\gamma \in \hat{G}$, describe an explicit basis for the space W_γ spanned by the highest weight vectors of the G-module $C^\infty_\gamma(S^2T^*_\mathbb{C})$.

(iii) For $\gamma \in \hat{G}$, consider the action of the Radon transform on the vectors of this basis for W_γ and prove that the inequality

$$\dim(\mathcal{N}_\mathbb{C} \cap W_\gamma) \leq \operatorname{Mult} C^\infty_\gamma(T_\mathbb{C})$$

holds whenever $\gamma \neq \gamma_1$, and that

$$\dim(\mathcal{N}_\mathbb{C} \cap W_{\gamma_1}) \leq \operatorname{Mult} C^\infty_{\gamma_1}(T_\mathbb{C}) - 1.$$

We used these methods to prove:

THEOREM 14. *The real Grassmannian $G^\mathbb{R}_{2,3}$ is rigid in the sense of Guillemin.*

For the real Grassmannian $G^\mathbb{R}_{2,3}$, the multiplicities of the G-modules $C^\infty_\gamma(S^2T^*_\mathbb{C})$ are ≤ 8.

Because all the Grassmannians $G^\mathbb{R}_{2,n}$, with $n \geq 3$, are of rank 2, this theorem implies that:

For all $n \geq 3$, the real Grassmannian $G_{2,n}^{\mathbb{R}}$ is rigid in the sense of Guillemin.

Now suppose that $X = G/K$ is one of the spaces admitting a non-zero invariant symmetric 3-form σ and is not equal to $SU(n)/S$. Let γ_0 and γ_1 be the elements of $\hat{G}$ corresponding to the trivial representation and to the adjoint representation of G on $\mathfrak{g}$, respectively. There is an involution τ of $\hat{G}$ such that $\tau(\gamma)$ is the contragredient representation of γ, for all $\gamma \in \hat{G}$.

The same methods could be used to prove the equality

$$(2) \quad \mathcal{N}_{\mathbb{C}} = D_\sigma(C^\infty(T_{\mathbb{C}}) \oplus C^\infty(E_{\mathbb{C}})) = D_0 C^\infty(T_{\mathbb{C}}) + \tilde{\sigma} d C^\infty(X).$$

Namely, it suffices to:

(i) Show that there exists an element $\gamma_2 \in \hat{G}$, with $\gamma_2 \neq \gamma_0, \gamma_1$, such that

$$D_0 C^\infty(T_{\mathbb{C}}) \cap \tilde{\sigma} d C^\infty(X) = \tilde{\sigma} d C^\infty_{\gamma_2}(X).$$

(ii) Prove that the operator D_σ is elliptic;

(iii) For all elements $\gamma \in \hat{G}$, determine the multiplicities of the G-modules $C^\infty_\gamma(X)$, $C^\infty_\gamma(T_{\mathbb{C}})$ and $C^\infty_\gamma(S^2 T^*_{\mathbb{C}})$.

(iv) Choose a subset Γ of $\hat{G}$ such that $\hat{G} = \Gamma \cup \tau(\Gamma)$; then, for all $\gamma \in \Gamma$, describe an explicit basis for the space W_γ spanned by the highest weight vectors of the G-module $C^\infty_\gamma(S^2 T^*_{\mathbb{C}})$.

(v) Consider the action of the Radon transform on the vectors of this basis for W_γ, with $\gamma \in \Gamma$, and prove that the inequality

$$\dim(\mathcal{N}_{\mathbb{C}} \cap W_\gamma) \leq \operatorname{Mult} C^\infty_\gamma(X) + \operatorname{Mult} C^\infty_\gamma(T_{\mathbb{C}})$$

holds for all $\gamma \in \Gamma$, with $\gamma \neq \gamma_0, \gamma_1, \gamma_2$, and that

$$\dim (\mathcal{N}_{\mathbb{C}} \cap W_{\gamma_2}) \leq \operatorname{Mult} C^\infty_{\gamma_2}(T_{\mathbb{C}}),$$

$$\mathcal{N}_{\mathbb{C}} \cap W_{\gamma_0} = \{0\}, \qquad \mathcal{N}_{\mathbb{C}} \cap W_{\gamma_1} = \{0\}.$$

We have done (i) when X is one of the spaces

$$SU(n)/SO(n), \qquad SU(2n)/Sp(n),$$

with $n \geq 3$. In this case, let $\mathcal{F}$ be the infinite dimensional subspace of $C^\infty_{\mathbb{R}}(X)$ such that

$$C^\infty_{\mathbb{R}}(X) = \mathcal{F} \oplus (C^\infty_{\gamma_2}(X) \cap C^\infty_{\mathbb{R}}(X))$$

is an orthogonal decomposition; note that the equality (2) then gives us the result of Theorem 10.

For the reduced space $X = Y_3$ of $SU(3)/SO(3)$, we have used these methods to prove the equality (2) and the result given by Theorem 10. Here the multiplicities of the $SU(3)$-modules $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$, and hence also the dimensions of the spaces W_γ, are ≤ 6.

For the space $SU(n)/S$, there is an analogous criterion which leads to the equality (2). When $n = 3$, we have used it to prove Theorem 10 for this space. For this space, the multiplicities of the modules $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$, and hence also the dimensions of the corresponding spaces W_γ, are again ≤ 6.

7. Differential operators and Guillemin rigidity

Let B be the sub-bundle of $\bigwedge^2 T^* \otimes \bigwedge^2 T^*$ of all curvature-like tensors on X. The curvature R of (X, g) is a section of B.

Let $\mathcal{R}'_g : C^\infty(S^2T^*) \to C^\infty(B)$ be the linearization of the curvature operator (along g). We consider a second-order differential operator

$$D_g = \mathcal{R}'_g + \kappa' : C^\infty(S^2T^*) \to C^\infty(B),$$

where $\kappa' : S^2T^* \to B$ is a certain morphism of vector bundles. This operator D_g has the following properties:

(i) If Y is a totally geodesic submanifold of X and $\iota : Y \to X$ is the natural embedding, and if $g_Y = \iota^* g$ is the induced metric on Y, we have

$$D_{g_Y} \iota^* h = \iota^* D_g h, \tag{3}$$

for all sections h of S^2T^*.

(ii) If (X, g) has constant curvature, then the Calabi sequence

$$C^\infty(T) \xrightarrow{D_0} C^\infty(S^2T^*) \xrightarrow{D_g} C^\infty(B) \tag{4}$$

is a complex.

Let $S^2_0 T^*$ be the sub-bundle of S^2T^* of symmetric 2-forms with zero trace and $\pi : S^2T^* \to S^2_0T^*$ be the orthogonal projection.

THEOREM 15. *The real projective space $\mathbb{RP}^n$ is rigid in the sense of Guillemin.*

PROOF: Let h be a symmetric 2-form satisfying the Guillemin condition. First, if (X, g) is the real projective plane $\mathbb{RP}^2$, then we have

$$C^\infty(S_0^2 T^*) = \pi D_0 C^\infty(T).$$

Thus we obtain

$$h = D_0 \xi + f g, \tag{5}$$

where ξ is a vector field on X and f is a real-valued function. Since $I_2(h) = 0$, we have $\hat{f} = 0$. Hence $f = 0$ and $h = D_0 \xi$. Next if $X = \mathbb{RP}^n$, using the rigidity of $\mathbb{RP}^2$ and assertions (i) and (ii), we obtain the equality

$$(D_g h)(\xi, \eta, \xi, \eta) = 0, \tag{6}$$

for all $\xi, \eta \in T$. This implies that $D_g h = 0$, and so by the exactness of the sequence (4), h is a Lie derivative of the metric.

THEOREM 16. *The complex projective space $\mathbb{CP}^n$ is rigid in the sense of Guillemin.*

PROOF: If $X = \mathbb{CP}^2$, then we consider the sub-bundle B^0 of B consisting of those tensors with zero trace. We have the decomposition

$$B^0 = B^0_+ \oplus B^0_-.$$

Let π_- be the orthogonal projection of B onto B^0_-. From twistor theory and the fact that $\pi_- W = 0$, where W is the Weyl tensor of X, we obtain the exactness of the sequence

$$C^\infty(T) \xrightarrow{\pi D_0} C^\infty(S_0^2 T^*) \xrightarrow{\pi_- D_g} C^\infty(B_-^0). \tag{7}$$

In fact, the operator is the linearization of the anti-dual Weyl tensor operator (along g). From the fact that the totally geodesic submanifolds of $\mathbb{CP}^2$ isometric to $\mathbb{RP}^2$ are rigid, we see that the equality (6) holds for all $\xi, \eta \in T$, whenever the complex planes generated by ξ and η are orthogonal. Algebraic computations show that this implies that

$$\pi_- D_g h = 0.$$

Thus by the exactness of (7), we see that (5) holds and the same argument as above gives us the rigidity of $\mathbb{CP}^2$. We obtain the rigidity of $\mathbb{CP}^n$ by an argument similar to the one used above.

Let ρ denote the representation of $T^* \otimes T$ on $\bigotimes^p T^*$. The infinitesimal orbit of the curvature

$$\tilde{B} = \{\, \rho(u)R \mid u \in T^* \otimes T,\ \rho(u)g = 0 \,\}$$

is a sub-bundle of B. Let $\alpha : B \to B/\tilde{B}$ be the natural projection.

The second-order differential operator

$$D_1 = \alpha \cdot D_g : C^\infty(S^2T^*) \to C^\infty(B/\tilde{B})$$

is part of the compatibility condition of the Killing operator D_0, i.e., we have a complex

$$C^\infty(T) \xrightarrow{D_0} C^\infty(S^2 T^*) \xrightarrow{D_1} C^\infty(B/\tilde{B}). \tag{8}$$

If the space X has constant curvature, then we have $\tilde{B} = \{0\}$ and we recover the Calabi sequence (4).

We consider the divergence operator

$$\operatorname{div} : C^\infty(S^2T^*) \to C^\infty(T^*),$$

which is essentially equal to the formal adjoint of the operator D_0. Since the operator D_0 is elliptic, the cohomology of this complex is isomorphic to the space

$$H(X) = \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0,\ D_1 h = 0 \,\}.$$

The space (X, g) is an Einstein manifold; the metric g satisfies

$$\operatorname{Ric}(g) = \lambda g,$$

where $\lambda > 0$. Thus we know that $\operatorname{Tr} \tilde{B} = \{0\}$, and so there is a well-defined trace mapping

$$\operatorname{Tr} : B/\tilde{B} \to S^2T^*.$$

We consider the linearization of the Ricci operator

$$\operatorname{Ric}'_g : C^\infty(S^2T^*) \to C^\infty(S^2T^*)$$

along g and the Lichnerowicz Laplacian

$$\Delta : C^\infty(S^2T^*) \to C^\infty(S^2T^*)$$

acting on symmetric 2-forms. In fact, if h is an element of $C^\infty(S^2T^*)$ satisfying $\operatorname{div} h = 0$, we have

$$\operatorname{Ric}'_g h = \frac{1}{2}\,(\Delta_L h - \operatorname{Hess}\operatorname{Tr} h). \tag{9}$$

By means of Lichnerowicz's Theorem concerning the first non-zero eigenvalue of the Laplacian of a compact Einstein manifolds with positive Ricci curvature, we obtain the following:

LEMMA 16. *Let N be a sub-bundle of B containing $\tilde{B}$ and E be a sub-bundle of S^2T^* satisfying*

$$\operatorname{Tr} N \subset E, \qquad \operatorname{Tr} E = \{0\}.$$

Let h be an element of $C^\infty(S^2T^*)$ satisfying

$$\operatorname{div} h = 0, \qquad D_1 h \in C^\infty(N/\tilde{B}).$$

Then we have

$$\operatorname{Tr} h = 0, \qquad \Delta h - 2\lambda h \in C^\infty(E).$$

Berger and Ebin introduced the (finite-dimensional) space of infinitesimal Einstein deformations

$$E(X) = \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0,\ \operatorname{Tr} h = 0,\ \Delta h = 2\lambda h \,\}$$

of the metric g.

If we take $N = \tilde{B}$ and $E = \{0\}$ in Lemma 16, we obtain the following:

Lemma 17. *The space $H(X)$ is finite-dimensional and is a subspace of $E(X)$.*

Theorem 18. *Let X be an irreducible symmetric space of compact type. If $E(X) = \{0\}$, then the sequence* (8) *is exact.*

According to Koiso, the Lichnerowicz Laplacian Δ is equal to the Casimir operator of the G-module $C^\infty(S^2T^*_{\mathbb{C}})$. From this fact, we obtain:

Proposition 19. *Suppose that X is not equal to a simple Lie group. Let γ_1 be the element of $\hat{G}$ which is the equivalence class of the irreducible G-module $\mathfrak{g}$. Then we have*

$$C^\infty_{\gamma_1}(S^2T^*_{\mathbb{C}}) = \{\, h \in C^\infty(S^2T^*_{\mathbb{C}}) \mid \Delta h = 2\lambda h \,\},$$

$$E(X) = \{\, h \in C^\infty_{\gamma_1}(S^2_0T^*_{\mathbb{C}}) \mid h = \bar{h},\ \operatorname{div} h = 0 \,\}.$$

Using this proposition, Koiso determines all the irreducible symmetric spaces of compact type whose infinitesimal Einstein deformations vanish. In particular, the space $E(X)$ is non-zero when X is the complex Grassmannian $G^{\mathbb{C}}_{m,n}$, with $m, n \geq 2$, or if $X = G/K$ admits a non-zero invariant symmetric 3-form. In fact, a non-zero G-invariant symmetric 3-form on X gives rise to an isomorphism between $E(X)$ and the Lie algebra of G.

8. Criterion for Guillemin rigidity

We shall now give a criterion for the Guillemin rigidity of X which exploits

(i) the fact that X is an Einstein manifold;

(ii) the hereditary properties of the operator D_1 with respect to totally geodesic submanifolds;

(iii) previously known results about Guillemin rigidity.

This criterion is used to prove

THEOREM 20. *For $m, n \geq 2$, with $m \neq n$, the Grassmannian $G^{\mathbb{K}}_{m,n}$ is rigid in the sense of Guillemin.*

We choose a family $\mathcal{F}'$ of closed connected totally geodesic submanifolds of X which are known to be rigid in the sense of Guillemin and a family $\mathcal{F}$ of closed connected totally geodesic surfaces of X each of which is contained in a submanifold belonging to $\mathcal{F}'$. Assume that the family $\mathcal{F}$ is invariant under the group G.

The set N consisting of those elements of B, which vanish when restricted to the closed totally geodesic submanifolds of $\mathcal{F}$, is a sub-bundle of B. The infinitesimal orbit of the curvature $\tilde{B}$ satisfies

$$\tilde{B} \subset N,$$

and we identify $N/\tilde{B}$ with a sub-bundle of $B/\tilde{B}$.

We denote by $\mathcal{L}(\mathcal{F}')$ the subspace of $C^\infty(S^2T^*)$ consisting of all symmetric 2-forms h satisfying the following condition: for all submanifolds $Z \in \mathcal{F}'$, the restriction of h to Z is a Lie derivative of the metric of Z induced by g. Then $D_0C^\infty(T)$ is a subspace of $\mathcal{L}(\mathcal{F}')$.

Using the vanishing of the infinitesimal orbits of the submanifolds belonging to $\mathcal{F}$, we obtain:

PROPOSITION 21. *A symmetric* 2*-form* h *on* X *belonging to* $\mathcal{L}(\mathcal{F}')$ *satisfies the relation* $D_1h \in C^\infty(N/\tilde{B})$.

Suppose that the family $\mathcal{F}'$ possesses the following property:

(P) If a section of S^2T^* over X satisfies the Guillemin condition, then its restriction to an arbitrary submanifold of X belonging to the family $\mathcal{F}'$ satisfies the Guillemin condition.

When there is a submanifold of $\mathcal{F}'$ whose rank is less than the rank of X, this property (P) is non-trivial.

If the family $\mathcal{F}'$ possesses property (P), then we see that

$$\mathcal{N} \subset \mathcal{L}(\mathcal{F}').$$

Our criterion for the Guillemin rigidity of the irreducible symmetric space X of compact type may be formulated as follows:

Theorem 22. *Let E be a G-sub-bundle of $S^2_0T^*$. Suppose that the family $\mathcal{F}'$ possesses property* (P) *and that the relations*

$$\operatorname{Tr} N \subset E, \qquad C^\infty(E) \cap \mathcal{L}(\mathcal{F}') = \{0\},$$

$$\mathcal{N} \cap E(X) = \{0\}$$

hold. Then the symmetric space X is rigid in the sense of Guillemin.

The proof of this theorem requires Lemma 16, Proposition 21, the harmonic analysis on X and Koiso's observation asserting that the Lichnerowicz Laplacian is equal to a Casimir operator.

This criterion, together with the rigidity of the projective spaces and the Grasmannian $G^{\mathbb{R}}_{2,3}$, is used to the rigidity of the Grassmannians given by Theorem 5.

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