Bäcklund Transformations and Darboux Integrability for Nonlinear Wave Equations

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Outline

A Motivating Example

Definitions & Results
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   Bäcklund transformations
   Main result

‘Backlund implies Darboux’
   $G$-structure for BTs between MA equations
   Implications of a BT to the wave equation

‘Darboux implies Bäcklund’
   G-structure for Darboux-integrable PDE
   EDS for BT to the wave equation
   Solving for a BT for particular equations

Open Problems
Two ways to solve Liouville’s equation $u_{xy} = e^u$

1. **Use Darboux integrability**: For any solution,

$$u_{xx} - \frac{1}{2}(u_x)^2 = f(x), \quad u_{yy} - \frac{1}{2}(u_y)^2 = g(y)$$

for some functions $f(x), g(y)$; conversely, given $f, g$, can solve these Riccati ODEs for $u_x, u_y$, and integrate to get $u$.

2. **Use a Bäcklund transformation (BT)**: Suppose $u(x, y)$ and $z(x, y)$ satisfy the system

$$u_x - z_x = 2 \exp((u + z)/2)$$
$$u_y + z_y = \exp((u - z)/2);$$

then $u$ satisfies Liouville iff $z$ satisfies $z_{xy} = 0$ (wave eqn.).

–set $z = F(x) + G(y)$ and solve compatible ODEs for $u$ in $x$- and $y$-directions

–both explicitly determine $u$ by solving ODE, given 2 functions

–what’s the relationship between these methods?
Darboux Integrability

**Defn** (Bryant-Griffiths-Hsu) A *hyperbolic EDS of class k* on $M^{k+4}$ is locally generated by 1-forms $\eta_0, \ldots, \eta_{k-1}$ and two independent decomposable 2-forms $\Omega_1, \Omega_2$ (whose factors, together with the $\eta$’s, form a coframe on $M$).

– special case for $k = 1$: a hyperbolic *Monge-Ampère* (MA) system on $M^5$, locally generated by $\Omega_1, \Omega_2$ and a contact form $\eta$

– prolongation of class $k$ hyperbolic EDS is hyperbolic of class $k + 2$

– define *characteristic system* $K_i, i = 1, 2$, spanned by the $\eta$’s and the factors of $\Omega_i$

**Defn** A hyperbolic EDS is *Darboux integrable* if $K_1$ and $K_2$ both contain Frobenius systems of rank 2, transverse to the span of the $\eta$’s, and Darboux *semi-integrable* if only one of the $K_i$ has this property.

– for $k = 1$, *Monge-integrable* means semi-integrable

– the only Darboux integrable hyperbolic MA system is the wave equation (up to contact equiv.)
Darboux Example: Liouville’s equation \( u_{xy} = e^u \)

–use Monge notation \( p = u_x, q = u_y, r = u_{xx}, s = u_{xy}, t = u_{yy} \)

–as a MA system on \( \mathbb{R}^5 \), generators are

\[
\eta = du - p
dx - q
dy,
\]

\[
\Omega_1 = (dp - e^u
dy) \wedge dx,
\]

\[
\Omega_2 = (dq - e^u
dx) \wedge dy
\]

–1st prolongation on \( \mathbb{R}^7 \) generated by \( \eta_0 = \eta \) and

\[
\eta_1 = dp - r
dx - e^u
dy,
\]

\[
\eta_2 = dq - e^u
dx - t
dy,
\]

\[
\Omega_1 = (dr - pe^u
dy) \wedge dx,
\]

\[
\Omega_2 = (dt - qe^u
dx) \wedge dy.
\]

–characteristic systems of the prolongation:

\[
K_1 = \{ \eta_0, \eta_1, \eta_2, dr - pe^u
dy, dx \} \supset \{ d(r - \frac{1}{2}p^2), dx \}
\]

\[
K_2 = \{ \eta_0, \eta_1, \eta_2, dt - qe^u
dx, dy \} \supset \{ d(t - \frac{1}{2}q^2), dy \}
\]

–any integral surface is foliated by integrals of \( K_1 \) and \( K_2 \)

–so, any solution has \( r - \frac{1}{2}p^2 = f(x) \) and \( t - \frac{1}{2}q^2 = g(y) \) for some \( f, g \)

–there are lots more examples; what about classification?
Classification of Darboux-Integrable MA Equations

Goursat (1898) classified the nonlinear MA equations of the form \( s = f(x, y, u, p, q) \) that are Darboux-integrable at 2-jet level (i.e., after one prolongation), up to contact transformations preserving the form:

(† means ‘Monge-integrable’)

\[
(x + y)s = 2\sqrt{pq}, \tag{I}
\]
\[
u s = \sqrt{1 + p^2}\sqrt{1 + q^2}, \tag{II}
\]
\[
(y + z)s = \sqrt{1 + p^2}\sqrt{1 + q^2}, \tag{III}
\]
\[
u s = \pm \phi(p)\psi(q), \quad \text{where } \phi(x), \psi(x) \text{ satisfy } df/dx \pm f/x = K \neq 0, \tag{IV}
\]
\[
(x + y)s = \alpha(p)\alpha(q), \quad \text{where } \alpha \text{ satisfies } \alpha(x) - 1 = \exp(x - \alpha(x)), \tag{V}
\]
\[
p - us/q = f(x, s/q), \tag{VI†}
\]
\[
s = e^{ut}\sqrt{1 + p^2}, \tag{VII}
\]
\[
p - y s = f(x, s), \tag{VIII†}
\]
\[
s = e^{ut}, \quad \text{(Liouville’s equation)} \tag{IX}
\]
\[
s = pe^{ut}, \tag{X†}
\]
\[
s = \left( (u - x)^{-1} + (u - y)^{-1} \right) pq. \tag{XI}
\]

Vessiot (1939-42) re-derived Goursat’s classification using Lie theory, showing that (VI) and (VIII) are contact-equivalent to respectively (X) and

\[
s = p/(x + y). \tag{VIII*}
\]
Classification, continued

Vessiot also classified linear Darboux-integrable equations, obtaining

\[ s = a(x, y)p + b(x, y)q - a(x, y)b(x, y)u, \] (XII)

where \( h(x, y) = -a_x \) and \( k(x, y) = -b_y \) must satisfy

\[ (\ln h)_{xy} = 2h - k, \quad (\ln k)_{xy} = 2k - h \] with \( h \neq k \), and

\[ s = 2u/(x + y)^2. \] (XIII)

–Juráš (2000) showed that any hyperbolic MA eqn that’s Darboux integrable at the 2-jet level must be contact-equivalent to the form \( s = f(x, y, u, p, q) \), hence on the Goursat-Vessiot list

–Biesecker (2004) re-derived the Goursat-Vessiot list using method of equivalence, characterizing each equation in terms of Laplace invariants
**Bäcklund transformations**

**Defn** (Goursat, 1925) A *Bäcklund transformation* between two hyperbolic MA systems, on $M$ and $\overline{M}$, is defined by a submfdld $B^6 \subset M \times \overline{M}$ such that:

![diagram](image)

- projections $\pi : B \to M$ and $\overline{\pi} : B \to \overline{M}$ are submersions
- pullbacks of generator 2-forms satisfy
  $$\{\Omega_1, \Omega_2\} \equiv \{\overline{\Omega}_1, \overline{\Omega}_2\} \equiv \{d\eta, d\overline{\eta}\} \mod \eta, \overline{\eta}$$

--corresponds to Goursat’s *type B*$_3$: if $\Sigma^2 \subset M$ is an integral of $\mathcal{I}$, then $\overline{\pi}^*\overline{\mathcal{I}}$ is Frobenius on $\pi^{-1}(\Sigma)$, giving a 1-parameter family of integrals for $\overline{\mathcal{I}}$, and vice-versa

--e.g., for Liouville’s equation on $M$ and the wave equation on $\overline{M}$ (with coords. $X, Y, Z, P, Q$), $B^6 \subset M \times \overline{M}$ is cut out by the equations

$$X = x, \quad Y = y, \quad p-P = 2 \exp((u+Z)/2), \quad q+Q = -\exp((u-Z)/2)$$
Constructing Bäcklund transformations

Remark Gardner (1978) shows how, for any class 3 hyperbolic system, to construct BT’s of type $B_1$ to some other PDE; e.g., from $s = e^u$ (Liouville) to $S = QZ$, by

$$X = x, \quad Y = y, \quad Z = p, \quad Q = e^u, P = r$$

Limit our attention to transformations of type $B_3$: for example,

Suppose $Q^4$ carries a hyperbolic EDS $\mathcal{J}$ of class 0 (locally, a quasilinear hyperbolic system of PDE for 2 fns. of 2 vars.), generated by $\Omega_1, \Omega_2$

–suppose $M, \bar{M}$ carry rank 1 integrable extensions of $\mathcal{J}$:

\begin{align*}
\begin{array}{ccc}
M & \xrightarrow{\rho} & Q \\
\downarrow & & \downarrow \\
B & \xrightarrow{\bar{\rho}} & \bar{M}
\end{array}
\end{align*}

–i.e., $d\eta \equiv 0 \mod \eta, \rho^* \Omega_1, \rho^* \Omega_2, \&$ same for $\bar{\eta}$

–let $B = \{(m, \bar{m}) \in M \times \bar{M} | \rho(m) = \bar{\rho}(\bar{m})\}$

–such BTs are called holonomic; less interesting
Our main result & its inspiration

—an result in Goursat (credited to Darboux) says that a 2nd-order PDE in the plane can be solved in terms of two arbitrary functions (and finitely many derivatives) by integrating ODE iff the given PDE is Darboux-integrable at some finite $k$-jet level

—Zvyagin (1991) published a classification of BTs between hyperbolic MA eqns. and the wave equation, omitting holonomics (and proof!)
—obtained 6 examples besides Liouville’s eqn., without giving the PDE in most cases (can identify 4 of them as (I), (II), (III) and (VII) )

Theorem (Clelland-I–) A hyperbolic MA equation has a BT to the wave equation if and only if it is Darboux-integrable at the 2-jet level.

—local in the ‘Darboux implies Bäcklund’ direction

—proof is independent of Goursat-Vessiot and Zvyagin classifications
G-structure for Bäcklund transformations

Clelland (2002): assume $\mathcal{I}_1, \mathcal{I}_2$ are hyperbolic MA systems on $\mathcal{M}_1, \mathcal{M}_2$, linked by a BT; then locally on $B^6 \ni$ a coframe $\theta_1, \theta_2, \omega_1, \omega_2, \omega_3, \omega_4$ such that $\pi_1^* \eta_1 = \theta_1$ and $\pi_2^* \eta_2 = \theta_2$ up to multiple, and

\[\begin{align*}
d\theta_1 &\equiv A_1 \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 \pmod{\theta_1}, \quad A_1 \neq 0, \\
d\theta_2 &\equiv \omega_1 \wedge \omega_2 + A_2 \omega_3 \wedge \omega_4 \pmod{\theta_2}, \quad A_2 \neq 0, \quad A_1 A_2 \neq 1, \\
d\omega_1 &\equiv B_1 \theta_1 \wedge \theta_2 + C_1 \omega_3 \wedge \omega_4 \pmod{\omega_1, \omega_2}, \\
d\omega_2 &\equiv B_2 \theta_1 \wedge \theta_2 + C_2 \omega_3 \wedge \omega_4 \pmod{\omega_1, \omega_2}, \\
d\omega_3 &\equiv B_3 \theta_1 \wedge \theta_2 + C_3 \omega_1 \wedge \omega_2 \pmod{\omega_3, \omega_4}, \\
d\omega_4 &\equiv B_4 \theta_1 \wedge \theta_2 + C_4 \omega_1 \wedge \omega_2 \pmod{\omega_3, \omega_4},
\end{align*}\]

–unique up to action of $G = GL(2) \times GL(2)$

–invariants of the BT are $A_1, A_2$ & vectors $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} B_3 \\ B_4 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$

–holonomic BTs have $B_1 = B_2 = B_3 = B_4 = 0$

–if $C_1 = C_2 = 0$ or $C_3 = C_4 = 0$, then both $\mathcal{I}_1$ and $\mathcal{I}_2$ encode the wave eqn.
“Bäcklund implies Darboux”: sketch of proof

–assume $M_1$ carries a MA system $\mathcal{I}_1$ contact-equivalent to the wave equation; then

$$\{\omega_1 - C_1\theta_1, \omega_2 - C_2\theta_1\} \quad \text{and} \quad \{\omega_3 - (C_3/A_1)\theta_1, \omega_4 - (C_4/A_1)\theta_1\}$$

are Frobenius systems (i.e., spanned by exact 1-forms)
–can set $A_1 = 1$, and assume that $C_2 \neq 0$, $C_4 \neq 0$; the above implies

$$d(C_1/C_2) \in J_1 = \{\theta_1, \theta_2, \omega_1, \omega_2\}, \quad d(C_3/C_4) \in J_2 = \{\theta_1, \theta_2, \omega_3, \omega_4\}.$$  

**Case A:** Both $J_1, J_2$ contain rank 3 Frobenius systems; then

$\{\theta_2, \omega_1, \omega_2\}$ and $\{\theta_2, \omega_3, \omega_4\}$ contain rank 2 Frob. systems, and $(M_2, \mathcal{I}_2)$ encodes the wave equation as well.

**Case B:** $J_2$ contains a rank 3 Frobenius system but $J_1$ doesn’t; again, $\{\theta_2, \omega_3, \omega_4\}$ contains rank 2 Frob. system, so $\mathcal{I}_2$ is Monge-integrable.
–need to check that the prolongation of $\mathcal{I}_2$ is Darboux-integrable
“Bäcklund implies Darboux”
Checking Darboux-integrability for Case B

Can set $C_1 = 0$; then $\theta_2, \omega_1$ live on $M_2$ but $\omega_2$ doesn’t.

Fake Argument: Pretend that EDS $\mathcal{I}_2$ on $M_2$ is generated by $\theta_2$, $\omega_1 \wedge \omega_2$ and $\omega_3 \wedge \omega_4$\n–on any integral surface with $\omega_1 \neq 0$, $\exists$ a function $r$ such that

$$\omega_2 - r\omega_1 = 0$$

–define (partial) prolongation of $\mathcal{I}_2$, on $M_2 \times \mathbb{R}$, generated by $\theta_2$ and $\psi = \omega_2 - r\omega_1$ (where $r$ is new coord.) & derivs.
–to check for Darboux-integrability, find $\pi = dr + \ldots$ such that

$$d\psi \equiv \omega_1 \wedge \pi \mod \theta_2, \psi$$

and see if $K = \{\theta_2, \psi, \pi, \omega_1\}$ contains a rank 2 Frob. system

Real Argument: Carry out the above, but replace $\omega_2$ by modified form $\tilde{\omega}_2 = e^a(\omega_2 - b\omega_1)$ that is well-defined on $M_2$. 
“Bäcklund implies Darboux”

Case C: Both \( J_1, J_2 \) contain only rank 2 Frob. systems

- set \( C_1 = C_3 = 0 \); then \( \theta_2, \omega_1, \omega_3 \) well-defined on \( M_2 \)
- using modified forms, define (full) prolongation of \( I_2 \) on \( M_2 \times \mathbb{R}^2 \)
generated by

\[
\psi_1 = \tilde{\omega}_2 - r\omega_1, \quad \psi_2 = \tilde{\omega}_4 - t\omega_3
\]

- find \( \pi_1, \pi_2 \) such that

\[
d\psi_1 \equiv \omega_1 \wedge \pi_1, \quad d\psi_2 \equiv \omega_3 \wedge \pi_2 \quad \text{ mod } \theta_2, \psi_1, \psi_2
\]

and check that the characteristic systems

\[
K_1 = \{ \theta_2, \psi_1, \pi_1, \omega_1 \}, \quad K_2 = \{ \theta_2, \psi_2, \pi_2, \omega_3 \}
\]
each contain a rank 2 Frobenius system.
“Darboux implies Bäcklund”: sketch of proof

Let $\mathcal{I}$ be a hyperbolic MA system on $M^5$, and $\hat{\mathcal{I}}$ its (full) prolongation on $N^7$; assume $\hat{\mathcal{I}}$ is Darboux-integrable.

Locally on $N$, $\exists$ a coframe $(\theta_0, \theta_1, \theta_2, \omega_1, \pi_1, \omega_2, \pi_2)$ such that

- pullback under $\rho : N \to M$ of the contact form is $\theta_0$, up to mult.
- $\hat{\mathcal{I}} = \langle \theta_0, \theta_1, \theta_2 \rangle_{\text{diff}}$, and

$$
\begin{align*}
d\theta_0 &\equiv \omega_1 \wedge \theta_1 + \omega_2 \wedge \theta_2 \mod \theta_0 \\
d\theta_1 &\equiv \omega_1 \wedge \pi_1 \mod \theta_0, \theta_1 \\
d\theta_2 &\equiv \omega_2 \wedge \pi_2 \mod \theta_0, \theta_2
\end{align*}
$$

- characteristic systems $K_1, K_2$ of $\hat{\mathcal{I}}$ contain Frobenius systems

$$
K_1^{(\infty)} = \{\omega_1, \pi_1\}, \quad K_2^{(\infty)} = \{\omega_2, \pi_2\}
$$

Main Case: Assume $\mathcal{I}$ is not Monge-integrable; then char. systems of $\mathcal{I}$ each contain unique (up to mult.) integrable 1-forms.

- adapt $\omega_1, \omega_2$ so that $d\omega_1 \equiv 0 \mod \omega_1, \quad d\omega_2 \equiv 0 \mod \omega_2$
- set of all such coframes gives a $G$-structure on $N$ with 5-d. fibers
For the $G$-str. there are connection forms $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ such that

$$d\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \pi_1 \\ \omega_2 \\ \pi_2 \end{bmatrix} = -\begin{bmatrix} \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma + \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma + \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & -\gamma - 2\alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_2 & -\gamma - 2\alpha_2 \end{bmatrix} \wedge \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \pi_1 \\ \omega_2 \\ \pi_2 \end{bmatrix}$$

choose $\omega_1, \omega_2$ to be exact, giving $\alpha_1 = D_1 \omega_1, \alpha_2 = D_2 \omega_2$

adjust $\pi_1, \pi_2$ so that

$$d\pi_1 = (\gamma - C_1 \theta_1 + E_1 \omega_1) \wedge \pi_1, \quad d\pi_2 = (\gamma - C_2 \theta_2 + E_2 \omega_2) \wedge \pi_2$$
A BT to the wave equation $Z_{XY} = 0$ would take the form of a bundle $B^6$ over $M$, with $Z$ as fiber coordinate:

$$
\begin{array}{cccc}
N^7 & B^6 & \pi \\
\rho & M^5 & \mathbb{R}^5 \\
\end{array}
$$

On the pullback of $B$ to $N$, the contact form for the wave equation would look like

$$
\Theta = dZ - P \omega_1 - Q \omega_2,
$$

where $P, Q$ are functions on $B$, such that

$$
dP \equiv P_1 \theta_1 \text{ mod } \theta_0, \omega_1, \Theta \\
dQ \equiv Q_1 \theta_2 \text{ mod } \theta_0, \omega_2, \Theta,
$$

$P_1, Q_1 \neq 0$.

– set up EDS on for functions $P, Q$ on $N \times \mathbb{R}$ (with $Z$ as new coord.)
– torsion absorption requires computing identities among second derivs. of $A_i, B_i, C_i, D_i, E_i$
– involutive at first prolongation, with $s_1 = 2$

**Monge-integrable case**: similar argument; BTs depend on $s_2 = 1$
Alternate Approach: Solving for a Bäcklund transformation

For specific equations on the Goursat-Vessiot list, it’s possible to solve PDE to get an explicit BT to the wave equation.

Example: Assume the BT for Liouville’s equation $s = e^u$ takes the form $p = f(u, x, y, Z, P, Q)$, $q = g(u, x, y, Z, P, Q)$. Then

$$(dp - e^u dy) \land dx \equiv ((f_y + qf_u + Qf_Z - e^u)dy + f_P dP + f_Q dQ) \land dx$$

mod contact forms, so that $f_Q = 0$, $f_y = e^u - gf_u - Qf_Z$; similarly $g_P = 0$ and $g_x = e^u - fg_u - Pg_Z$.

Other PDEs arise from requiring that

$$r - \frac{1}{2}p^2 = f_x + ff_u + Pf_Z + Rf_P - \frac{1}{2}f^2$$ = a function of $x, P, R$ only

$$t - \frac{1}{2}q^2 = g_y + gg_u + Qg_Z + Tg_Q - \frac{1}{2}g^2$$ = a function of $y, Q, T$ only,

implying that

$$f_{Pu} = f_{PZ} = 0, \quad \partial_Z, \partial_u, \partial_y(f_x + ff_u + Pf_Z - \frac{1}{2}f^2) = 0,$$

$$g_{Qu} = g_{QZ} = 0, \quad \partial_Z, \partial_u, \partial_x(g_y + gg_u + Qg_Z - \frac{1}{2}g^2) = 0.$$
Solving for BT’s: Example, continued

Differentiating and equating mixed partials yields more equations:

\[ f_z = -g_Q f_u, \quad g_z = -f_P g_u, \quad f_P x = 0, \quad g_Q y = 0, \]

then

\[ 2g_u f_u = e^u, \quad g_Q = -f_P = \text{constant}. \]

–system is now involutive, again with \( s_1 = 2 \)

–setting \( f_P = 1 \), solve these equations to get the BT

\[
\begin{align*}
p &= P + 2 \exp((u + z + v(x) + w(y))/2) + v'(x) \\
q &= -Q + \exp((u - z - v(x) - w(y))/2) - w'(y)
\end{align*}
\]

where \( v(x) \) and \( w(y) \) are arbitrary functions

–using this technique shows BTs to the wave equation exist for (IV), (V), (IX), (XI), (XII) and (XIII); can obtain some explicit examples

–for (XI), (XII) and (XIII) all such BTs are holonomic, with \( s_1 = 2 \)

–technique not feasible for Monge-integrable (VI), (VIII), (X)
Open Problems

- Relate these Bäcklund transformations to *nonlinear superposition formulas* (Anderson, Fels, Vassiliou)
- Identify remaining Zvyagin examples using Biesecker’s characterizations
- Characterize special types of BTs: quasilinear, variational, and especially *auto-Bäcklund transformations*
- Completely integrable PDE (‘soliton’ equations) often have BTs containing an arbitrary parameter, e.g., for sine-Gordon

\[
\begin{align*}
    v_x &= u_x + \frac{\lambda}{2} \sin((u + v)/2) \\
    v_y &= -u_y - \frac{1}{2\lambda} \sin((u - v)/2).
\end{align*}
\]

\(\lambda \neq 0\).

- characterize those BTs where a parameter can be inserted (e.g., by lifting a symmetry from one side).