Symmetry Reduction And The Method of Darboux

Exterior Differential Systems and the Method of Equivalence – MSRI

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May 11, 2008

Overview

Theory

- A Generalization of the Notion of Hyperbolic EDS
- A General Definition of Darboux Integrablity
- Symmetry Reduction of Differential Systems
- Vessiot's Method for Creating Darboux Integrable Equations
- Examples
- Theorem

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Applications

- A Classical Example -Revisited.
- A Group Theoretical Approach To Internal Equivalences
- The Cauchy Problem By Quadratures
- Moutard's Theorem Revisited

Decomposable EDS

Generalize the idea of a hyperbolic Pfaffian system to include:

$$\text{systems} \begin{cases} u_{xy} &= A(x, u, u, v, u_x, u_y, v_x, v_y) \\ v_{xy} &= B(x, u, u, v, u_x, u_y, v_x, v_y) \end{cases}$$

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$$\text{3rd order}\begin{cases} u_{xz} &= A(x, u, z, u, u_x, u_{yz}, u_{zz}) \\ u_{xzz} &= B(x, u, z, u, u_x, u_y, u_z, u_{yz}, u_{zz}, u_{yyz}, u_{zzz}) \end{cases}$$

Decomposable EDS

Definition

$I = \{\,\boldsymbol{\theta}\,\}$ is **decomposable** if there is a (multi-colored) coframe

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with block diagonal structure equations

$$\begin{bmatrix} d\,\boldsymbol{\theta} \\ d\,\boldsymbol{\theta} \\ d\,\boldsymbol{\theta} \end{bmatrix} \equiv \begin{bmatrix} *\boldsymbol{\pi}\wedge\boldsymbol{\pi} & 0 \\ 0 & *\boldsymbol{\pi}\wedge\boldsymbol{\pi} \\ 0 & 0 \end{bmatrix} \quad \text{mod } \{\boldsymbol{\theta}\}$$

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Put

$$V = \{\boldsymbol{\theta}, \boldsymbol{\pi}\} \quad V = \{\boldsymbol{\theta}, \boldsymbol{\pi}\}.$$

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This definition goes well beyond the classical *integration by ODE definition* .

Vessiot's Method for Creating DI EDS

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quotient differential system

=

reduction of DE using differential invariants

Examples

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Categorize group actions according to the kinds of differential equations they produce under reduction.

Here are some examples:

Example I

Action: Abelian Intransitive Actions on R^2, $J^k(R, R) \times J^l(R, R)$.

Equation Type: Linear

Example I

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Equation Type: Linear

Action	k	l	Equation
$\partial_\phi, \alpha\partial_\phi, \alpha^2\partial_\phi$	3	3	$u_{xy} + \dfrac{2u}{(x-y)^2} = 0$
$\partial_\phi, \alpha\partial_\phi, \ldots \alpha^4\partial_\phi$	5	5	$u_{xy} + \dfrac{6u}{(x-y)^2} = 0$
$\partial_\phi, \alpha\partial_\phi, \ldots, \alpha^4\partial_\phi$	4	6	$u_{xy} + \dfrac{-2u_y}{(x-y)} + \dfrac{4u}{(x-y)^2} = 0$
$\partial_\phi, \alpha\partial_\phi, \ldots \alpha^4\partial_\phi$	3	7	$u_{xy} + \dfrac{-2u_y}{(x-y)} = 0$
$\partial_\phi, \cos(\alpha)\partial_\phi, \sin(\alpha)\partial_\phi$	3	3	$u_{xy} + \dfrac{u}{1-\cos(x-y)} = 0$

Example II

Action: Solvable Intransitive Actions on R^2, $J^k(R, R) \times J^l(R, R)$.

Equation Type: Moutard (linearizable)

$$u_{xy} + \partial_x(Ae^u) + \partial_y(Be^{-u}) + C = 0$$

Example II

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Equation Type: Moutard (linearizable)

$$u_{xy} + \partial_x(Ae^u) + \partial_y(Be^{-u}) + C = 0$$

Action	k	l	Equation
$\partial_\phi, \alpha\partial_\phi, \phi\partial_\phi$	3	3	$A = -B = \dfrac{1}{x-y}, C = 0$
$\partial_\phi, \alpha\partial_\phi, \alpha^2\partial_\phi, \phi\partial_\phi$	4	5	$A = -2B = \dfrac{2}{x-y}, C = \dfrac{-1}{(x-y)^2}$
$\partial_\phi, \dfrac{2}{\alpha}\partial_\phi, \phi\partial_\phi$	3	3	$A = \dfrac{2x}{(x-y)}, B = \dfrac{-1}{x(x-y)}$ $C = \dfrac{-1}{(x-y)^2}$

Example III

Action: $\mathfrak{sl}2$ Intransitive Actions on R^2, $J^k(R, R) \times J^l(R, R)$.

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Equation Type: Moutard (non-linearizable)

Action	k	l	Equation
$\partial_\phi, \phi\partial_\phi, \phi^2\partial_\phi$	3	3	$u_{xy} = e^u$
$\partial_\phi, \phi\partial_\phi, \phi^2\partial_\phi$	2	4	$u_{xy} + u_x u = 0$

Example IV

Action: Abelian subgroups of g_2 on the Hilbert-Cartan equation $\psi' = (\phi'')^2$

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Francesco Strazzullo:

$$\mathfrak{g}_2 = \{Y_6, Y_5, Y_4, Y_3, Y_2, Y_1, H_1, H_2, X_1, X_2, X_3, X_4, X_5, X_6\}$$

Action	Equation
Y_1, X_4, X_5	$rt - s^2 = st^{1/3}$
X_4, X_5, X_6	$rt - s^2 = 3t^4$
X_2, X_5, X_6	$rt - s^2 = \frac{-4t^2s^2}{s^3+4t^2}$
[illegible]	[illegible]

Example V

Action: Imprimitive actions on R^2 $J^l(R^2, R) \times J^k(R^2, R)$

Equation Type: Triangular wave map systems

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Action	Equation
GLKO[14]	$u_{xy} = e^u, v_{xy} - n(n+1)e^{2u}v = 0$
GLKO[25]	$u_{xy} = u_y e^u, v_{xy} + ((n-\alpha)e^u + \alpha u_x)v_x = 0$
GLKO[26]	$u_{xy} = u_y e^u,$ $v_{xy} - e^u v_y + (n+1_u)yv_x + (n+1_)!e^u u_y = 0$
GLKO[27]	$u_{xy} = u_x e^u,$ $v_{xy} + (e^v)_x - (nu_x e^{u-v})_y + (n-2)e^u u_x = 0$

Example VI

Action: Primitive actions on R^2 $J^l(R^2, R) \times J^k(R^2, R)$

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Diego Calalano

$$\begin{cases} u_{xy} &= \dfrac{uu_xu_y + vu_yv_x - uv_xv_y + uu_y + vu_xv_y + vv_y}{u^2+v^2} \\ v_{xy} &= \dfrac{vv_xv_y + uu_xv_y + uu_yv_x + uv_y - vu_xv_y - vu_y}{u^2+v^2} \end{cases}$$

$$\begin{cases} u_{xy} &= \dfrac{uv + v_y}{v} u_x \\ v_{xy} &= \dfrac{uv_y + u_xv^2 + v_xv_y - u_yv}{v} \end{cases}$$

$$\begin{cases} u_{xy} &= \dfrac{v_xv_y - u_xu_y}{2e^u + 2} \\ v_{xy} &= -\dfrac{u_xv_y + u_yv_x}{2e^u + 2}. \end{cases}$$

$$\begin{cases} u_{xy} &= 2e^u - 2e^v \\ v_{xy} &= -e^u + 2e^v \end{cases}$$

A Group Theoretical Classification of DI EDS

Theorem [IA, Mark Fels, Peter Vassiliou]

Let I be a Darboux integrable Pfaffian system.

- Let J be the pullback of I to an integral manifold of V.
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Then there is a (algorithmically computed) group action G on M such that

$$I = (J + J)/G.$$

A famous example of Goursat

$$u_{xy} = 2n\frac{\sqrt{u_x u_y}}{x+y}$$

Here are the invariants for small values of n(where $p = sqrt(u_x)$) :

$$I_2 = \frac{2p}{(x+y)^2} + \frac{4p_x}{(x+y)} + p_{xx}$$

$$I_3 = \frac{6p}{(x+y)^3} + \frac{18p_x}{(x+y)^2} + \frac{9p_{xx}}{(x+y)} + p_{xxx}$$

$$I_4 = \frac{24p}{(x+y)^4} + \frac{96p_x}{(x+y)^3} + \frac{72p_{xx}}{(x+y)^2} + \frac{16p_{xxx}}{(x+y)} + p_{xxxx}$$

$$I_5 = \frac{120p}{(x+y)^5} + \frac{600p_x}{(x+y)^4} + \frac{600p_{xx}}{(x+y)^3} + \frac{200p_{xxx}}{(x+y)^2} + \frac{25p_{xxxx}}{x+y} + p_{xxxxx}$$

Some invariants for J

The equation $u_{xy} = 2n\dfrac{\sqrt{u_x u_y}}{x+y}$ is Darboux integrable on at the $n+1$-jet level so that we set this equation up as a rank 2n+1 Pfaffian I system on a manifold M of dimension $2n+5$.

In accordance with the general theory, consider the restriction J of I to the level set W of the invariants, W is of dimension $2n+3$.

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Explicit formulas for this Pfaffian system are rather complicated so that it is not apparent what the canonical form for this Pfaffian system is ....

Invariants

so we first compute some invariants.

n	J_n	W_n	J_n^-	W_n^-	Derived Flag	Sym	Levi
	rk	dm	rk	dm	Growth	dim	ss · rad
2	5	7	3	5	[2,1,2]	14	$\mathfrak{g}_2$
3	7	9	4	6	[2,1,2 1]	11	$\mathfrak{sl}_2 \cdot [8, 7, 1, 0]$
4	9	11	5	7	[2,1,2 1, 1]	13	$\mathfrak{sl}_2 \cdot [10, 9, 1, 0]$
5	10	12	6	8	[2,1,2 1, 1, 1]	15	$\mathfrak{sl}_2 \cdot [12, 11, 1, 0]$

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A result of Dubrov and Zelenko and state that only the Pfaffian systems J with these symbol algebras are precisely

$$u' = (v'')^n.$$

This then allows us to explicitily integration the Goursat equation.

n=3

$$u = \frac{1}{2}(X+Y) + \frac{1}{\gamma^6} \cdot \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ V_0 \\ V_1 \\ V_2 \end{bmatrix}^t \begin{bmatrix} -720 & 360 & -60 & -720 & 360 & -60 \\ 360 & -192 & 36 & 360 & -168 & 24 \\ -60 & 36 & -9 & -60 & 24 & -3 \\ -720 & 360 & -60 & -720 & 360 & -60 \\ 360 & -168 & 24 & 360 & -192 & 36 \\ -60 & 24 & -3 & -60 & 36 & -9 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ V_0 \\ V_1 \\ V_2 \end{bmatrix},$$

where $\gamma = x + y$, $U_k = \gamma^{(n-k)} \dot{U}^{(k)}$, $V_k = \gamma^{(n-k)} \dot{V}^{(k)}$ and

$$\frac{dX}{dx} = \frac{d^n U}{dx^n}^2 \quad \text{and} \quad \frac{dY}{dy} = \frac{d^n V}{dy^n}^2.$$

The Cauchy Problem for $F(x, y, u, p, q, r, s, t) = 0$

Here are solutions to the Cauchy problem for some classical examples.

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$$3u_{xx}u_{yy}^3 + 1 = 0 \quad x = 0, \quad y = \epsilon, \quad u(0,\epsilon) = \epsilon^2 \quad u_x(0,\epsilon) = 0.$$

$$x = 2\delta - 2\epsilon \quad y = \frac{\epsilon + \delta}{2}, \quad u = \frac{\delta^2 + \epsilon^2 + 4\epsilon\delta}{6}$$

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$$u_{xy} - uu_x = 0 \quad x = \epsilon, \quad y = \epsilon, \quad u(\epsilon, \epsilon) = \epsilon, \quad u_x(\epsilon, \epsilon) = 1.$$

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$$u = \text{Maple wallpaper involving} \int \frac{\text{BesselK}(3/4, \epsilon^2/2)}{\text{BesselK}(1/4, \epsilon^2/2)}$$

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The Vessiot group is $SL2$ so one expects to see a Ricatti equation somewhere and indeed

$$R = \frac{\text{BesselK}(3/4, \epsilon^2/2)}{\text{BesselK}(1/4, \epsilon^2/2)} \quad \text{solves} \quad R' = -\frac{\epsilon}{2} + \frac{R}{\epsilon} + \frac{R^2}{2}$$

Transformation Theory

Our main theorem has important applications for the transformation theory of DI systems. Suppose we have two DI Pfaffian systems I and K on 7-manifolds with the and Vessiot group G and such that

$$\begin{array}{ccc} J^3 \times J^3 & \longrightarrow & J^2 \times J^4 \\ {\scriptstyle G_3}\big\downarrow & & {\scriptstyle G_3}\big\downarrow \\ M_7,\ I & \longrightarrow & Q_7,\ K \end{array}$$

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Partially prolong:

$$\begin{array}{ccc} J^3 \times J^4 & \xrightarrow{=} & J^3 \times J^4 \\ {\scriptstyle G_3}\big\downarrow & & {\scriptstyle G_3}\big\downarrow \\ M_8,\ I^{(0,1)} & \xrightarrow{\Phi} & Q_8,\ K^{(1,0)} \end{array}$$

The map Φ is a diffeomorhism which gives an interal equivalence between the partial prolonged differential systems $I^{(0,1)}$ and $K^{(1,0)}$.

Examples

I.

$$u_{xy} + \frac{2u}{(x-y)^2} = 0 \quad U_{XY} - \frac{4U_Y}{(X-Y)} = 0$$

$$U = (\frac{1}{2}(x-y)^2 u_y \quad \text{with inverse} \quad u = U_X - \frac{2}{x-y}U$$

Note: This formulas should be prolonged by differentiation.

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$$u_{xy} + u^2 u_{yy} + 2uu_y^2 = 0 \quad U_{XY} = 2\sqrt{U_X U_Y}/(X+Y)$$

$$x = X, \quad y = U + (X+Y)U_Y, \quad u = \sqrt{U_X} + \sqrt{U_Y}$$

Generalization of Moutard

On November 8, 1908 Forysth gave the Presidential Address to the Cambridge Mathematical Society. This address contains an nice summary of classical geometric integration methods and concludes with a number of open problems. One of these is to classify all 2nd order scalar PDE in the plane whose general integral is

$$x = V_1(\alpha, \beta, \phi(\alpha), \psi(\beta)\phi', \psi', \dots),$$
$$y = V_2(\alpha, \beta, \phi(\alpha), \psi(\beta)\phi', \psi', \dots),$$
$$z = V_3(\alpha, \beta, \phi(\alpha), \psi(\beta)\phi', \psi', \dots).$$

Conjecture : These are precisely the equations obtained by quotients of jet spaces and therefore this open PDE classical problem reduces to classification of contact transforms on $J^1(R, R)$, a problem first solved by Lie.

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