Two small goals: Problems of Lie

Two small goals: Problems of Lie

Lie 1882:

Lie 1882:

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

English translation: A vector field is projective w.r.t. a metric, if its flow takes (unparametrized) geodesics to geodesics.

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

English translation: A vector field is projective w.r.t. a metric, if its flow takes (unparametrized) geodesics to geodesics.

S. Lie 1882: Describe all 2 dim metrics admitting

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

English translation: A vector field is projective w.r.t. a metric, if its flow takes (unparametrized) geodesics to geodesics.

S. Lie 1882: Describe all 2 dim metrics admitting

- *Problem I: one projective vector field*

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

English translation: A vector field is projective w.r.t. a metric, if its flow takes (unparametrized) geodesics to geodesics.

S. Lie 1882: Describe all 2 dim metrics admitting

- *Problem I: one projective vector field*
- *Problem II: many projective vector fields*

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

English translation: A vector field is projective w.r.t. a metric, if its flow takes (unparametrized) geodesics to geodesics.

S. Lie 1882: Describe all 2 dim metrics admitting

- *Problem I: one projective vector field*
- *Problem II: many projective vector fields*

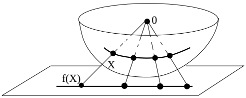
Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

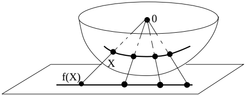
Both problems are local, in a neighborhood of a generic point, pseudoriemannian metrics are allowed

Examples of Lagrange 1789

Examples of Lagrange 1789

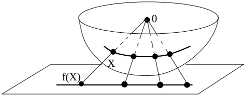


Examples of Lagrange 1789



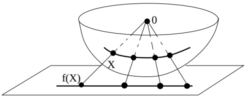
Radial projection $f : S^2 \longrightarrow \mathbb{R}^2$

Examples of Lagrange 1789



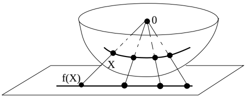
Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane,

Examples of Lagrange 1789



Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

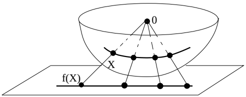
Examples of Lagrange 1789



Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

Thus, for every Killing vector field on the plane ist pullback is a projective vector field on the sphere

Examples of Lagrange 1789

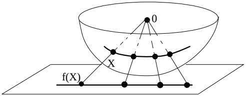


Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

Thus, for every Killing vector field on the plane ist pullback is a projective vector field on the sphere

Motivation of Lagrange:

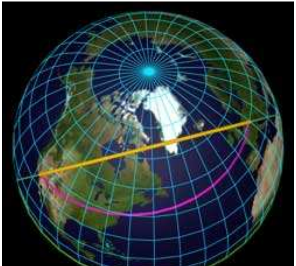
Examples of Lagrange 1789



Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

Thus, for every Killing vector field on the plane ist pullback is a projective vector field on the sphere

Motivation of Lagrange:



Solution of the 2nd Lie Problem

Solution of the 2nd Lie Problem

Theorem (Bryant, Manno, M$\sim$ 2007)

Solution of the 2nd Lie Problem

Theorem (Bryant, Manno, M∼ 2007) *If a two-dimensional metric g of nonconstant curvature has at least 2 projective vector fields*

Solution of the 2nd Lie Problem

Theorem (Bryant, Manno, M$\sim$ 2007) *If a two-dimensional metric g of nonconstant curvature has at least 2 projective vector fields such that they are linear independent at the point p,*

Solution of the 2nd Lie Problem

Theorem (Bryant, Manno, M$\sim$ 2007) *If a two-dimensional metric g of nonconstant curvature has at least 2 projective vector fields such that they are linear independent at the point p, then there exist coordinates x, y in a neighborhood of p such that the metrics are as follows.*

Solution of the 2nd Lie Problem

Theorem (Bryant, Manno, M$\sim$ 2007) *If a two-dimensional metric g of nonconstant curvature has at least 2 projective vector fields such that they are linear independent at the point p, then there exist coordinates x, y in a neighborhood of p such that the metrics are as follows.*

1. $\epsilon_1 e^{(b+2)\,x} dx^2 + \epsilon_2 b e^{b\,x} dy^2$, *where* $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\epsilon_i \in \{-1, 1\}$
2. $a\left(\epsilon_1 \frac{e^{(b+2)\,x} dx^2}{(e^{b\,x}+\epsilon_2)^2} + \frac{e^{b\,x} dy^2}{e^{b\,x}+\epsilon_2}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\epsilon_i \in \{-1, 1\}$
3. $a\left(\frac{e^{2\,x} dx^2}{x^2} + \epsilon \frac{dy^2}{x}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, *and* $\epsilon \in \{-1, 1\}$
4. $\epsilon_1 e^{3x} dx^2 + \epsilon_2 e^{x} dy^2$, *where* $\epsilon_i \in \{-1, 1\}$,
5. $a\left(\frac{e^{3x} dx^2}{(e^{x}+\epsilon_2)^2} + \frac{\epsilon_1 e^{x} dy^2}{(e^{x}+\epsilon_2)}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $\epsilon_i \in \{-1, 1\}$,
6. $a\left(\frac{dx^2}{(cx+2x^2+\epsilon_2)^2 x} + \epsilon_1 \frac{x dy^2}{cx+2x^2+\epsilon_2}\right)$, *where* $a > 0$, $\epsilon_i \in \{-1, 1\}$, $c \in \mathbb{R}$.

Example for 1st Problem of Lie: infinitesimal homotheties

Example for 1st Problem of Lie: infinitesimal homotheties

Def. A vector field is an infinitesimal homothety , if its flow multiply the metric by a constant (depending on the time).

Example for 1st Problem of Lie: infinitesimal homotheties

Def. A vector field is an infinitesimal homothety , if its flow multiply the metric by a constant (depending on the time).

Example. $\frac{\partial}{\partial x} = (1, 0)$ is an infinitesimal homothety for the metric $e^{\lambda x}\left(E(y)dx^2 + F(y)dxdy + G(y)dy^2\right)$.

Example for 1st Problem of Lie: infinitesimal homotheties

Def. A vector field is an infinitesimal homothety, if its flow multiply the metric by a constant (depending on the time).

Example. $\frac{\partial}{\partial x} = (1, 0)$ is an infinitesimal homothety for the metric $e^{\lambda x}\left(E(y)dx^2 + F(y)dxdy + G(y)dy^2\right)$.

Every infinitesimal homothety is a projective vector field.

Example for 1st Problem of Lie: infinitesimal homotheties

Def. A vector field is an infinitesimal homothety, if its flow multiply the metric by a constant (depending on the time).

Example. $\frac{\partial}{\partial x} = (1,0)$ is an infinitesimal homothety for the metric $e^{\lambda x}\left(E(y)dx^2 + F(y)dxdy + G(y)dy^2\right)$.

Every infinitesimal homothety is a projective vector field.

Def: Two metrics g and $\bar{g}$ (on one manifold) are geodesically equivalent if they have the same unparametrized geodesics

Example for 1st Problem of Lie: infinitesimal homotheties

Def. A vector field is an infinitesimal homothety, if its flow multiply the metric by a constant (depending on the time).

Example. $\frac{\partial}{\partial x} = (1, 0)$ is an infinitesimal homothety for the metric $e^{\lambda x}\left(E(y)dx^2 + F(y)dxdy + G(y)dy^2\right)$.

Every infinitesimal homothety is a projective vector field.

Def: Two metrics g and $\bar{g}$ (on one manifold) are geodesically equivalent if they have the same unparametrized geodesics

Of cause, if v is projective w.r.t. g, then it is projective w.r.t. every geodesically equivalent $\bar{g}$

Example for 1st Problem of Lie: infinitesimal homotheties

Def. A vector field is an infinitesimal homothety , if its flow multiply the metric by a constant (depending on the time).

Example. $\frac{\partial}{\partial x} = (1, 0)$ is an infinitesimal homothety for the metric $e^{\lambda x} \left(E(y)dx^2 + F(y)dxdy + G(y)dy^2\right)$.

Every infinitesimal homothety is a projective vector field.

Def: Two metrics g and $\bar{g}$ (on one manifold) are geodesically equivalent if they have the same unparametrized geodesics

Of cause, if v is projective w.r.t. g, then it is projective w.r.t. every geodesically equivalent $\bar{g}$

For explicitly given metric g, it is possible (and relatively easy with the help of Maple) to describe the space of all geodesically equivalent metric $\bar{g}$:
Shulikovsky 1954 – Kruglikov 2007 – Bryant-Dunajski-Eastwood 2008

Theorem (M$\sim$ 2008):

Theorem (M$\sim$ 2008): *Let v be a projective vector field on $(M^2, \bar{g})$. Assume the restriction of g to no neighborhood has an infinitesimal homothety. Then, there exists a coordinate system in a neighborhood of almost every point such that certain metric g geodesically equivalent to $\bar{g}$ is given by*

Theorem (M∼ 2008): *Let v be a projective vector field on $(M^2, \bar{g})$. Assume the restriction of g to no neighborhood has an infinitesimal homothety. Then, there exists a coordinate system in a neighborhood of almost every point such that certain metric g geodesically equivalent to $\bar{g}$ is given by*

1. $ds_g^2 = (X(x) - Y(y))(X_1(x)dx^2 + Y_1(y)dy^2), v = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, sodass
   - 1.1 $X(x) = \frac{1}{x}, \ \ Y(y) = \frac{1}{y}, X_1(x) = C_1 \cdot \frac{e^{-3x}}{x}, \ Y_1(y) = \frac{e^{-3y}}{y}$.
   - 1.2 $X(x) = \tan(x), \ Y(y) = \tan(y), \ X_1(x) = C_1 \cdot \frac{e^{-3\lambda x}}{\cos(x)}$, $Y_1(y) = \frac{e^{-3\lambda y}}{\cos(y)}$.
   - 1.3 $X(x) = C_1 \cdot e^{\nu x}, \ Y(y) = e^{\nu y}, \ X_1(x) = e^{2x}, \ Y_1(y) = \pm e^{2y}$.
2. $ds_g^2 = (Y(y) + x)dxdy, \ v = v_1(x,y)\frac{\partial}{\partial x} + v_2(y)\frac{\partial}{\partial y}$, sodass
   - 2.1 $Y = e^{\frac{3}{2y}} \cdot \frac{\sqrt{y}}{y-3} + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2} d\xi$, $v_1 = \frac{y-3}{2}\left(x + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2} d\xi\right), \ v_2 = y^2$.
   - 2.2 $Y = e^{-\frac{3}{2}\lambda \arctan(y)} \cdot \frac{\sqrt[4]{y^2+1}}{y-3\lambda} + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi$, $v_1 = \frac{y-3\lambda}{2}\left(x + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi\right), \ v_2 = y^2 + 1$.
   - 2.3 $Y(y) = y^{\nu}, \ v_1(x,y) = \nu x, \ v_2 = y$.

It is time to start the talk

Vladimir Matveev
Jena (Germany)

Vladimir Matveev
Jena (Germany)

Quadratic integrals for geodesics flows and projective vector fields

Vladimir Matveev
Jena (Germany)

Quadratic integrals for geodesics flows and projective vector fields

www.minet.uni-jena.de/∼matveev/

Definition

Definition

A function $F : T^*M \longrightarrow \mathbb{R}$ is called **an integral** of the geodesic flow of g,

Definition

A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$,

Definition

A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2}g^{ij}p_i p_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric.

Definition

A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2} g^{ij} p_i p_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric.
The integral F is **quadratic in momenta**

Definition

A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2} g^{ij} p_i p_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric.
The integral F is **quadratic in momenta** if, in every local coordinate system (x, y) on M^2, it has the form

$$a(x, y) p_x^2 + b(x, y) p_x p_y + c(x, y) p_y^2.$$

Definition

A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2} g^{ij} p_i p_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric.
The integral F is **quadratic in momenta** if, in every local coordinate system (x, y) on M^2, it has the form

$$a(x, y)p_x^2 + b(x, y)p_x p_y + c(x, y)p_y^2. \tag{1}$$

The PDE for the function (1) to be an integral for the metric $ds_g^2 = f(x, y)dxdy$ is

Definition

A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2} g^{ij} p_i p_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric.
The integral F is **quadratic in momenta** if, in every local coordinate system (x, y) on M^2, it has the form

$$a(x,y)p_x^2 + b(x,y)p_xp_y + c(x,y)p_y^2. \tag{1}$$

The PDE for the function (1) to be an integral for the metric $ds_g^2 = f(x,y)dxdy$ is

$$\begin{cases} a_y &= 0\,, \\ f\,a_x + f\,b_y + 2f_x a + f_y b &= 0\,, \\ f\,b_x + f\,c_y + f_x b + 2f_y c &= 0\,, \\ c_x &= 0\,. \end{cases} \tag{2}$$

Definition

A function $F : T^*M \to \mathbb{R}$ is called **an integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2} g^{ij} p_i p_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric.
The integral F is **quadratic in momenta** if, in every local coordinate system (x, y) on M^2, it has the form

$$a(x,y)p_x^2 + b(x,y)p_x p_y + c(x,y)p_y^2. \qquad (1)$$

The PDE for the function (1) to be an integral for the metric $ds_g^2 = f(x,y)dxdy$ is

$$\begin{cases} a_y = 0\,, \\ f\,a_x + f\,b_y + 2f_x a + f_y b = 0\,, \\ f\,b_x + f\,c_y + f_x b + 2f_y c = 0\,, \\ c_x = 0\,. \end{cases} \qquad (2)$$

We see that the system is overdetermined and of finite type.

Why to study quadratic integrals ?

Why to study quadratic integrals ?

- Because they are classical

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...)

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

.

- Because they are useful for description of metrics with the same geodesics:

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

.

- Because they are useful for description of metrics with the same geodesics:

  Theorem (Dini,Darboux $< - - - - - -- >$ Topalov, M$\sim$ 1998)
  $g \sim \bar{g}$

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

.

- Because they are useful for description of metrics with the same geodesics:

  Theorem (Dini,Darboux $< - - - - - -- >$ Topalov, M$\sim$ 1998)
  $g \sim \bar{g}$ iff the function

$$I : TM^2 \to \mathbb{R},$$

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

.

- Because they are useful for description of metrics with the same geodesics:

  Theorem (Dini,Darboux $< - - - - - -- >$ Topalov, M$\sim$ 1998) $g \sim \bar{g}$ iff the function

$$I : TM^2 \to \mathbb{R}, \quad I(\xi) := \bar{g}(\xi, \xi) \left(\frac{\det(g)}{\det(\bar{g})} \right)^{2/3}$$

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

.

- Because they are useful for description of metrics with the same geodesics:

Theorem (Dini,Darboux $< - - - - - -- >$ Topalov, M$\sim$ 1998) $g \sim \bar g$ iff the function

$$I : TM^2 \to \mathbb{R}, \quad I(\xi) := \bar g(\xi, \xi) \left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3}$$

is an integral of the geodesic flow of g.

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

.

- Because they are useful for description of metrics with the same geodesics:

Theorem (Dini,Darboux $< - - - - - -- >$ Topalov, M$\sim$ 1998)
$g \sim \bar g$ iff the function

$$I : TM^2 \to \mathbb{R}, \quad I(\xi) := \bar g(\xi,\xi)\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3}$$

is an integral of the geodesic flow of g. We identify TM and T^*M by g.

What people studied about quadratic integrals?

What people studied about quadratic integrals?

- Local description/classification.

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\,dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x)-Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1+xY'(y))\,dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869		

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869	Its complexification	

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M$\sim$08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x) - Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)} p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)} p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M∼08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals)

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M∼08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals) (Koenigs 1896,

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M∼08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals) (Koenigs 1896, Winternitz 1969

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h) dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x) - Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)} p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)} p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M∼08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals) (Koenigs 1896, Winternitz 1969 , Miller, Kalnins, Kress (nowdays)).

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M∼08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals) (Koenigs 1896, Winternitz 1969 , Miller, Kalnins, Kress (nowdays)).

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M∼08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals) (Koenigs 1896, Winternitz 1969 , Miller, Kalnins, Kress (nowdays)).

Theorem (Koenigs 1896, Lie 1882 $< - - - >$ Bryant, Manno, M∼07)

The space of quadratics integrals is $\geq$ 4-dimensional	$\Longleftrightarrow$	the space of projective vector fields is $\geq$ 3-dimensional

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\, dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)} p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)} p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M∼08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals) (Koenigs 1896, Winternitz 1969 , Miller, Kalnins, Kress (nowdays)).

Theorem (Koenigs 1896, Lie 1882 $< - - - >$ Bryant, Manno, M∼07)

The space of quadratics integrals is $\geq$ 4-dimensional	$\Longleftrightarrow$	the space of projective vector fields is $\geq$ 3-dimensional

We will use it for Lie Problems

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

Is nonlinear

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi, \xi) \left(\frac{\det(g)}{\det(\bar{g})} \right)^{2/3}$

PDE for "$\bar g$ is geodesically equivalent to a given g"

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar g$ then $I(\xi) := \bar g(\xi, \xi) \left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3}$ is an integral

PDE for "$\bar{g}$ is geodesically equivalent to a given g"

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi, \xi) \left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ is an integral
- Second observation:

PDE for "$\bar{g}$ is geodesically equivalent to a given g"

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi, \xi)\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ is an integral
- Second observation: I is integral

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi, \xi)\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I, H\}_g = 0$

PDE for “$\bar g$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar g$ then $I(\xi) := \bar g(\xi,\xi)\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I, H\}_g = 0 \iff \{\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3} \bar g(\xi,\xi), H(\xi)\}_g = 0$

PDE for “$\bar g$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar g$ then $I(\xi) := \bar g(\xi,\xi)\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I, H\}_g = 0 \iff$
  $\{\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3} \bar g(\xi,\xi), H(\xi)\}_g = 0$
  The last equation is linear in $\bar g/\det(\bar g)^{2/3}$.

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi,\xi)\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I, H\}_g = 0 \iff$
  $\{\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3} \bar{g}(\xi,\xi), H(\xi)\}_g = 0$
  The last equation is linear in $\bar{g}/\det(\bar{g})^{2/3}$.

Moreover, the equation is projectively invariant

PDE for “$\bar{g}$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi,\xi)\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I,H\}_g = 0 \iff$
  $\{\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3} \bar{g}(\xi,\xi), H(\xi)\}_g = 0$
  The last equation is linear in $\bar{g}/\det(\bar{g})^{2/3}$.

Moreover, the equation is projectively invariant : if we replace g by any other geodesically equivalent metric $\hat{g}$

PDE for "$\bar{g}$ is geodesically equivalent to a given g"

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi, \xi) \left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I, H\}_g = 0 \iff$
  $\{\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3} \bar{g}(\xi, \xi), H(\xi)\}_g = 0$
  The last equation is linear in $\bar{g}/\det(\bar{g})^{2/3}$.

Moreover, the equation is projectively invariant : if we replace g by any other geodesically equivalent metric $\hat{g}$, the equation will not be changed

PDE for “$\bar g$ is geodesically equivalent to a given g“

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar g$ then $I(\xi) := \bar g(\xi,\xi)\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I,H\}_g = 0 \iff$
  $\{\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3}\bar g(\xi,\xi), H(\xi)\}_g = 0$
  The last equation is linear in $\bar g/\det(\bar g)^{2/3}$.

Moreover, the equation is projectively invariant : if we replace g by any other geodesically equivalent metric $\hat g$, the equation will not be changed .

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\, K_0\, a_{12} - 2/3\, K_1\, a_{11} & = & 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\, K_0\, a_{22} + 2/3\, K_1\, a_{12} - 4/3\, K_2\, a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\, K_1\, a_{22} - 2/3\, K_2\, a_{12} - 2\, K_3\, a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\, K_2\, a_{22} - 2\, K_3\, a_{12} & = & 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11},\ K_1 := \Gamma^1_{11} - 2\Gamma^2_{12},\ K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12},\ K_3 = \Gamma^1_{22}$.

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\,K_0\,a_{12} - 2/3\,K_1\,a_{11} &= 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\,K_0\,a_{22} + 2/3\,K_1\,a_{12} - 4/3\,K_2\,a_{11} &= 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\,K_1\,a_{22} - 2/3\,K_2\,a_{12} - 2\,K_3\,a_{11} &= 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\,K_2\,a_{22} - 2\,K_3\,a_{12} &= 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11}$, $K_1 := \Gamma^1_{11} - 2\Gamma^2_{12}$, $K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12}$, $K_3 = \Gamma^1_{22}$.

Important observation:

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\left\{\begin{array}{rcl} \frac{\partial a_{11}}{\partial x} + 2\,K_0 a_{12} - 2/3\,K_1 a_{11} & = & 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\,K_0 a_{22} + 2/3\,K_1 a_{12} - 4/3\,K_2 a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\,K_1 a_{22} - 2/3\,K_2 a_{12} - 2\,K_3 a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\,K_2 a_{22} - 2\,K_3 a_{12} & = & 0 \end{array}\right.$$

where $K_0 = -\Gamma^2_{11}$, $K_1 := \Gamma^1_{11} - 2\Gamma^2_{12}$, $K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12}$, $K_3 = \Gamma^1_{22}$.

Important observation: The system of PDE is projectively invariant: it does not depend on the connection within connections with the same geodesics.

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\, K_0\, a_{12} - 2/3\, K_1\, a_{11} & = & 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\, K_0\, a_{22} + 2/3\, K_1\, a_{12} - 4/3\, K_2\, a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\, K_1\, a_{22} - 2/3\, K_2\, a_{12} - 2\, K_3\, a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\, K_2\, a_{22} - 2\, K_3\, a_{12} & = & 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11}$, $K_1 := \Gamma^1_{11} - 2\Gamma^2_{12}$, $K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12}$, $K_3 = \Gamma^1_{22}$.

Important observation: The system of PDE is projectively invariant: it does not depend on the connection within connections with the same geodesics.

PDE-background of the observation:

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\,K_0\,a_{12} - 2/3\,K_1\,a_{11} & = & 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\,K_0\,a_{22} + 2/3\,K_1\,a_{12} - 4/3\,K_2\,a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\,K_1\,a_{22} - 2/3\,K_2\,a_{12} - 2\,K_3\,a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\,K_2\,a_{22} - 2\,K_3\,a_{12} & = & 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11}$, $K_1 := \Gamma^1_{11} - 2\Gamma^2_{12}$, $K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12}$, $K_3 = \Gamma^1_{22}$.

Important observation: The system of PDE is projectively invariant: it does not depend on the connection within connections with the same geodesics.

PDE-background of the observation: K_i determine unparameterized geodesics.

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\,K_0\,a_{12} - 2/3\,K_1\,a_{11} & = & 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\,K_0\,a_{22} + 2/3\,K_1\,a_{12} - 4/3\,K_2\,a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\,K_1\,a_{22} - 2/3\,K_2\,a_{12} - 2\,K_3\,a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\,K_2\,a_{22} - 2\,K_3\,a_{12} & = & 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11}$, $K_1 := \Gamma^1_{11} - 2\Gamma^2_{12}$, $K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12}$, $K_3 = \Gamma^1_{22}$.

Important observation: The system of PDE is projectively invariant: it does not depend on the connection within connections with the same geodesics.

PDE-background of the observation: K_i determine unparameterized geodesics.

Geometric background of observation:

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\, K_0 a_{12} - 2/3\, K_1 a_{11} & = & 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\, K_0 a_{22} + 2/3\, K_1 a_{12} - 4/3\, K_2 a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\, K_1 a_{22} - 2/3\, K_2 a_{12} - 2\, K_3 a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\, K_2 a_{22} - 2\, K_3 a_{12} & = & 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11}$, $K_1 := \Gamma^1_{11} - 2\Gamma^2_{12}$, $K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12}$, $K_3 = \Gamma^1_{22}$.

Important observation: The system of PDE is projectively invariant: it does not depend on the connection within connections with the same geodesics.

PDE-background of the observation: K_i determine unparameterized geodesics.

Geometric background of observation: The integrals for g allow to construct integrals for $\bar{g}$, if $g \sim \bar{g}$,

The linear equation on $a := \bar g/\det(\bar g)^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\,K_0 a_{12} - 2/3\,K_1 a_{11} & = & 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\,K_0 a_{22} + 2/3\,K_1 a_{12} - 4/3\,K_2 a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\,K_1 a_{22} - 2/3\,K_2 a_{12} - 2\,K_3 a_{11} & = & 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\,K_2 a_{22} - 2\,K_3 a_{12} & = & 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11},\ K_1 := \Gamma^1_{11} - 2\Gamma^2_{12},\ K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12},\ K_3 = \Gamma^1_{22}$.

Important observation: The system of PDE is projectively invariant: it does not depend on the connection within connections with the same geodesics.

PDE-background of the observation: K_i determine unparameterized geodesics.

Geometric background of observation: The integrals for g allow to construct integrals for $\bar g$, if $g \sim \bar g$, because g and $\bar g$ have the same geodesics.

How we solved Lie Problems

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g).

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields.

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.

We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v \binom{\sigma_1}{\sigma_2}$

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v\binom{\sigma_1}{\sigma_2} = \mathrm{B}\binom{\sigma_1}{\sigma_2}$,

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$,
where B is a 2×2 matrix

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$,
where B is a 2×2 matrix .
By choice of basis we can make
B

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B}\binom{\sigma_1}{\sigma_2}$,
where B is a 2×2 matrix .
By choice of basis we can make
$\mathrm{B} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$,

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \mathrm{B} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$,
where B is a 2×2 matrix .
By choice of basis we can make
$\mathrm{B} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, or $\begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}$,

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$,
where B is a 2×2 matrix .
By choice of basis we can make
$$\mathrm{B} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v\binom{\sigma_1}{\sigma_2}$

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v\binom{\sigma_1}{\sigma_2} = \mathrm{B}\binom{\sigma_1}{\sigma_2}$

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v\binom{\sigma_1}{\sigma_2} = \mathrm{B}\binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v\binom{\sigma_1}{\sigma_2} = \mathrm{B}\binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have 9 cases to consider.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v\binom{\sigma_1}{\sigma_2} = \mathrm{B}\binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have 9 cases to consider.
- In every case the data in the normal form of (σ_1, σ_2), i.e., the functions $X(x)$, $Y(y)$ for Liouville case, h for complex-liouville case, $Y(y)$ for Jordan-block case, have at most two first derivatives.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v\binom{\sigma_1}{\sigma_2} = \mathrm{B}\binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have 9 cases to consider.
- In every case the data in the normal form of (σ_1, σ_2), i.e., the functions $X(x)$, $Y(y)$ for Liouville case, h for complex-liouville case, $Y(y)$ for Jordan-block case, have at most two first derivatives.
- Thus, the equation $L_v\binom{\sigma_1}{\sigma_2}$

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have 9 cases to consider.
- In every case the data in the normal form of (σ_1, σ_2), i.e., the functions $X(x)$, $Y(y)$ for Liouville case, h for complex-liouville case, $Y(y)$ for Jordan-block case, have at most two first derivatives.
- Thus, the equation $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \mathrm{B} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have 9 cases to consider.
- In every case the data in the normal form of (σ_1, σ_2), i.e., the functions $X(x)$, $Y(y)$ for Liouville case, h for complex-liouville case, $Y(y)$ for Jordan-block case, have at most two first derivatives.
- Thus, the equation $L_v \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \mathrm{B} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ contains 6 PDE and 6 highest derivatives of the unknown functions. i.e., is a Frobenius system.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have 9 cases to consider.
- In every case the data in the normal form of (σ_1, σ_2), i.e., the functions $X(x)$, $Y(y)$ for Liouville case, h for complex-liouville case, $Y(y)$ for Jordan-block case, have at most two first derivatives.
- Thus, the equation $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ contains 6 PDE and 6 highest derivatives of the unknown functions. i.e., is a Frobenius system.
- Such systems can be solved by hands. We did it and solved the Lie Problems.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have 9 cases to consider.
- In every case the data in the normal form of (σ_1, σ_2), i.e., the functions $X(x)$, $Y(y)$ for Liouville case, h for complex-liouville case, $Y(y)$ for Jordan-block case, have at most two first derivatives.
- Thus, the equation $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ contains 6 PDE and 6 highest derivatives of the unknown functions. i.e., is a Frobenius system.
- Such systems can be solved by hands. We did it and solved the Lie Problems.

If somebody did not manage to follow

Solution of Lie problem is based on

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals
3. Jordan normal forms for matrices

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals
3. Jordan normal forms for matrices

These allowed to simplify the equations: instead of 4 equations of the second order which immediately come out,

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals
3. Jordan normal forms for matrices

   These allowed to simplify the equations: instead of 4 equations of the second order which immediately come out, we construct 6 equations of the first order.

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals
3. Jordan normal forms for matrices

   These allowed to simplify the equations: instead of 4 equations of the second order which immediately come out, we construct 6 equations of the first order. The price we paid: there are 9 different systems

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals
3. Jordan normal forms for matrices

   These allowed to simplify the equations: instead of 4 equations of the second order which immediately come out, we construct 6 equations of the first order. The price we paid: there are 9 different systems
4. solving 9 Frobenius systems.

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals
3. Jordan normal forms for matrices

   These allowed to simplify the equations: instead of 4 equations of the second order which immediately come out, we construct 6 equations of the first order. The price we paid: there are 9 different systems

4. solving 9 Frobenius systems.

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):**

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field.*

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler		

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	Solodovnikov (1956) proved the conjecture

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	Solodovnikov (1956) proved the conjecture assuming that all objects are real analytic

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	Solodovnikov (1956) proved the conjecture assuming that all objects are real analytic

Global questions: Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	Solodovnikov (1956) proved the conjecture assuming that all objects are real analytic and that $n \geq 3$.

Proof of Lichnerowich conjecture

Proof of Lichnerowich conjecture

Proof of Lichnerowich conjecture

(**Lichnerowicz Conjecture:** *Among closed Riemannian manifolds, only the round sphere admits essential projective vector fields*)

Proof of Lichnerowich conjecture

(**Lichnerowicz Conjecture:** *Among closed Riemannian manifolds, only the round sphere admits essential projective vector fields*)

The multidimensional version of Liouville equations

The multidimensional version of Liouville equations

Sinjukov 1962/

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 /

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M$\sim$ 2003.

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M$\sim$ 2003.
Theorem (Eastwood, M$\sim$ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M$\sim$ 2003.

Theorem (Eastwood, M$\sim$ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\mathrm{Tracefreepartof}\left(\nabla_a \sigma^{bc}\right) = 0.$$

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M$\sim$ 2003.
Theorem (Eastwood, M$\sim$ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\mathrm{Trace free part of} \left(\nabla_a \sigma^{bc}\right) = 0.$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2M \otimes (\Lambda_n)^{2/(n+1)} M$.

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M$\sim$ 2003.
Theorem (Eastwood, M$\sim$ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\mathrm{Trace free part of} \left(\nabla_a \sigma^{bc}\right) = 0.$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2 M \otimes (\Lambda_n)^{2/(n+1)} M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a}\sigma^{bc} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1}\Gamma^d_{da}\,\sigma^{bc}}_{\text{addition coming from volume form}}$$

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M∼ 2003.

Theorem (Eastwood, M∼ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\text{Trace free part of}\left(\nabla_a \sigma^{bc}\right) = 0.$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2 M \otimes (\Lambda_n)^{2/(n+1)} M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a}\sigma^{bc} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1}\Gamma^d_{da}\,\sigma^{bc}}_{\text{addition coming from volume form}}$$

ADVANTAGES:

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M∼ 2003.

Theorem (Eastwood, M∼ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\mathrm{Tracefreepartof}\left(\nabla_a \sigma^{bc}\right) = 0.$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2M \otimes (\Lambda_n)^{2/(n+1)}M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a}\sigma^{bc} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1}\Gamma^d_{da}\,\sigma^{bc}}_{\text{addition coming from volume form}}$$

ADVANTAGES:

1. It is a LINEAR PDE-system of the first order.

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M$\sim$ 2003.

Theorem (Eastwood, M$\sim$ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\mathrm{Tracefreepartof}\left(\nabla_a \sigma^{bc}\right) = 0.$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2 M \otimes (\Lambda_n)^{2/(n+1)} M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a}\sigma^{bc} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1}\Gamma^d_{da}\,\sigma^{bc}}_{\text{addition coming from volume form}}$$

ADVANTAGES:

1. It is a LINEAR PDE-system of the first order.
2. The system does not depend on the choice of Γ within a projective class.

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M$\sim$ 2003.

Theorem (Eastwood, M$\sim$ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\mathrm{Tracefreepartof}\left(\nabla_a \sigma^{bc}\right) = 0.$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2M \otimes (\Lambda_n)^{2/(n+1)}M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a}\sigma^{bc} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1}\Gamma^d_{da}\sigma^{bc}}_{\text{addition coming from volume form}}$$

ADVANTAGES:

1. It is a LINEAR PDE-system of the first order.
2. The system does not depend on the choice of Γ within a projective class. (short tensor calculations)
3. For dim2 it is the Liouville system we used to solve the Lie problems.

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then,

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$,

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $: \mathcal{A} \longrightarrow \mathcal{A}$.

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $: \mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v \binom{\sigma_1}{\sigma_2} =$

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\binom{\sigma_1}{\sigma_2} = \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}\binom{\sigma_1}{\sigma_2}$, or

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, or $L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, or $L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$
depending on the eigenvalues of L_v.
In the left case

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\binom{\sigma_1}{\sigma_2} = \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}\binom{\sigma_1}{\sigma_2}$, or $L_v\binom{\sigma_1}{\sigma_2} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\binom{\sigma_1}{\sigma_2}$
depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get

$$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}, \quad \text{or} \quad L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$$

depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$ implying v is homothety vector for g

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\binom{\sigma_1}{\sigma_2} = \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}\binom{\sigma_1}{\sigma_2}$, or $L_v\binom{\sigma_1}{\sigma_2} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\binom{\sigma_1}{\sigma_2}$
depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$ implying v is homothety vector for g which is impossible for closed manifolds.

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, or $L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$
depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$ implying v is homothety vector for g which is impossible for closed manifolds. In the second case

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, or $L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$
depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$ implying v is homothety vector for g which is impossible for closed manifolds. In the second case the determinant of σ vanishes at some point

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, or $L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$
depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$ implying v is homothety vector for g which is impossible for closed manifolds. In the second case the determinant of σ vanishes at some point (because $\cos(t)$ vanishes for some t)

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, or $L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$
depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$ implying v is homothety vector for g which is impossible for closed manifolds. In the second case the determinant of σ vanishes at some point (because $\cos(t)$ vanishes for some t) implying that $g = \sigma \cdot \det(\sigma)$ vanishes somewhere.

Proof of Lichnerowicz conjecture if $dim(\mathcal{A}) = 2$

Denote by $\mathcal{A}$ the space of solutions.
Assume first $\mathcal{A}$ is two-dimenional.
Every projective vector field v does not change the Liouville-Sinjukov-Eastwood-Matveev equations.
Then, for every solution $\sigma \in \mathcal{A}$, we have: $L_v\sigma$ is also a soluiton. Thus, L_v is a linear mapping $:\mathcal{A} \longrightarrow \mathcal{A}$.

Then, for the appropiate choice of the basis σ_1, σ_2 we get
$L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & *\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, or $L_v\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$
depending on the eigenvalues of L_v.
In the left case there exists a solution s.t. $L_v\sigma = \alpha\sigma$ implying v is homothety vector for g which is impossible for closed manifolds. In the second case the determinant of σ vanishes at some point (because $\cos(t)$ vanishes for some t) implying that $g = \sigma \cdot \det(\sigma)$ vanishes somewhere. Lichnerowich conjecture is proved under assumption that the space of solutions is two-dimenional

What happens if the space of solutions is more than two-dimensional?

What happens if the space of solutions is more than two-dimensional?

Theorem (Matveev 2004) *Assume* Γ *is the Levi-Civita connection of a Riemannian metric on a closed or complete manifold.*

What happens if the space of solutions is more than two-dimensional?

Theorem (Matveev 2004) *Assume* Γ *is the Levi-Civita connection of a Riemannian metric on a closed or complete manifold. If* $\dim(\mathcal{A}) \geq 3$, *then* g *has constant curvature.*

What happens if the space of solutions is more than two-dimensional?

Theorem (Matveev 2004) *Assume* Γ *is the Levi-Civita connection of a Riemannian metric on a closed or complete manifold. If* $\dim(\mathcal{A}) \geq 3$, *then* g *has constant curvature.*

Initial proof is complicated.

Explanation using newer results

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008)

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$. *Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij}/det(g)^{1/(n+1)}$ *satisfy the equation*

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$ *. Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij}/det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \tag{3}$$

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$ *. Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij} / det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \tag{3}$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equations

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$. *Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij}/det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2 f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \tag{3}$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equations .

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$. *Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij} / det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2 f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \qquad (3)$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equations .

(3) is a famous equation: it appears everywhere in local differential geometry:

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$ *. Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij} / det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2 f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \tag{3}$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equations .

(3) is a famous equation: it appears everywhere in local differential geometry:

- Solodovnokov 1956: in projective transformations

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$. *Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij}/det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2 f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \qquad (3)$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equations .

(3) is a famous equation: it appears everywhere in local differential geometry:

- Solodovnokov 1956: in projective transformations
- deVries 1953: in conformal transformations

Explanation using newer results

Theorem (Bolsinov, Kiosak, M∼ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$. *Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij} / det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2 f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \tag{3}$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M∼ equations .

(3) is a famous equation: it appears everywhere in local differential geometry:

- Solodovnokov 1956: in projective transformations
- deVries 1953: in conformal transformations
- Mikes-Kiosak 2003: in Einstein manifolds

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$. *Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij} / det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2 f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \qquad (3)$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equations .

(3) is a famous equation: it appears everywhere in local differential geometry:

- Solodovnokov 1956: in projective transformations
- deVries 1953: in conformal transformations
- Mikes-Kiosak 2003: in Einstein manifolds
- Obata – Tanno 1970: eigenfunction of Laplace equation on the round sphere corresponding to second eigenvalue satisfy this equation.

Explanation using newer results

Theorem (Bolsinov, Kiosak, M$\sim$ 2008) *Let* $\dim(\mathcal{A}) \geq 3$ *and* $\dim(M) \geq 3$ *. Then, for every solution* σ *its trace* $f := \sigma^{ij} g_{ij}/det(g)^{1/(n+1)}$ *satisfy the equation (for a certain constant* K*).*

$$f_{,ijk} = K \cdot (2 f_k g_{ij} + f_i g_{jk} + f_j g_{ik}). \qquad (3)$$

Method of Proof: We prolonged two times the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equations .

(3) is a famous equation: it appears everywhere in local differential geometry:

- Solodovnokov 1956: in projective transformations
- deVries 1953: in conformal transformations
- Mikes-Kiosak 2003: in Einstein manifolds
- Obata – Tanno 1970: eigenfunction of Laplace equation on the round sphere corresponding to second eigenvalue satisfy this equation.

Theorem of Tanno 1970 finishes the proof:

Theorem of Tanno 1970 finishes the proof:

Theorem (Tanno 1970 for positive K; Kiosak 2003 $< -- >$ M$\sim$ 2004 for all K)

Theorem of Tanno 1970 finishes the proof:

Theorem (Tanno 1970 for positive K**; Kiosak 2003** $<-->$ **M**$\sim$ **2004 for all** K**)** *Let* $(M^{n\geq 3}, g)$ *be a closed Riemannian manifold*

Theorem of Tanno 1970 finishes the proof:

Theorem (Tanno 1970 for positive K; Kiosak 2003 $<-->$ M$\sim$ 2004 for all K) *Let $(M^{n\geq 3}, g)$ be a closed Riemannian manifold such that there exists a nontrivial solution of the equation (3):*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

Theorem of Tanno 1970 finishes the proof:

Theorem (Tanno 1970 for positive K**; Kiosak 2003** $<-->$ **M**$\sim$ **2004 for all** K**)** *Let* $(M^{n\geq 3}, g)$ *be a closed Riemannian manifold such that there exists a nontrivial solution of the equation (3):*

$$f_{,ijk} = K\cdot(2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

Then, g *has constant positive curvature.*

Theorem of Tanno 1970 finishes the proof:

Theorem (Tanno 1970 for positive K; Kiosak 2003 $< -- >$ M$\sim$ 2004 for all K) *Let $(M^{n\geq 3}, g)$ be a closed Riemannian manifold such that there exists a nontrivial solution of the equation (3):*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

Then, g has constant positive curvature.

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008:

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008: *A complete pseudoriemannian manifold of dim ≥ 3 does not admit a complete essential projective vector field.*

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008: *A complete pseudoriemannian manifold of dim ≥ 3 does not admit a complete essential projective vector field.*

Joint project with Bolsinov and Kiosak:

Scheme of the possible proof/way to find counterexample.

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008: *A complete pseudoriemannian manifold of dim* ≥ 3 *does not admit a complete essential projective vector field.*

Joint project with Bolsinov and Kiosak:

Scheme of the possible proof/way to find counterexample.

- Let $\mathcal{A}$ be the space of solutions of the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equation.

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008: *A complete pseudoriemannian manifold of dim ≥ 3 does not admit a complete essential projective vector field.*

Joint project with Bolsinov and Kiosak:

Scheme of the possible proof/way to find counterexample.

- Let $\mathcal{A}$ be the space of solutions of the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equation. If $\mathcal{A}$ is two-dimensional, then, as in the solution of S. Lie problem

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008: *A complete pseudoriemannian manifold of dim ≥ 3 does not admit a complete essential projective vector field.*

Joint project with Bolsinov and Kiosak:

Scheme of the possible proof/way to find counterexample.

- Let $\mathcal{A}$ be the space of solutions of the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equation. If $\mathcal{A}$ is two-dimensional, then, as in the solution of S. Lie problem , one can describe all such metrics

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008: *A* complete *pseudoriemannian manifold of dim ≥ 3 does not admit a complete essential projective vector field.*

Joint project with Bolsinov and Kiosak:

Scheme of the possible proof/way to find counterexample.

- Let $\mathcal{A}$ be the space of solutions of the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equation. If $\mathcal{A}$ is two-dimensional, then, as in the solution of S. Lie problem , one can describe all such metrics – one need to understand whether they could be complete.

Lichnerowich conjecture in the pseudo-Riemannian case: Work in Progress:

Conjecture M$\sim$ 2008: *A complete pseudoriemannian manifold of dim ≥ 3 does not admit a complete essential projective vector field.*

Joint project with Bolsinov and Kiosak:

Scheme of the possible proof/way to find counterexample.

- Let $\mathcal{A}$ be the space of solutions of the Liouville-Sinjukov-Bolsinov-Eastwood-M$\sim$ equation. If $\mathcal{A}$ is two-dimensional, then, as in the solution of S. Lie problem , one can describe all such metrics – one need to understand whether they could be complete.

- If $dim(\mathcal{A}) \geq 3$

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008)

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let* $(M^{n\geq 3}, g)$ *be a complete pseudoriemannian manifold*

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let* $(M^{n\geq 3}, g)$ *be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3)*

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M∼ April 2008) *Let* $(M^{n\geq 3}, g)$ *be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M∼ April 2008) *Let* $(M^{n\geq 3}, g)$ *be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ *geodesically equivalent to* g *has the same Levi-Civita connection with* g*.*

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M∼ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.
- **Corollary:**

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.
- **Corollary: Complete Einstein manifolds are geodesically rigid:**

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M∼ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.
- **Corollary: Complete Einstein manifolds are geodesically rigid:** *Let (M, g) is a complete pseudoriemannian Einstein manifold.*

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

- **Corollary: Complete Einstein manifolds are geodesically rigid:** *Let (M, g) is a complete pseudoriemannian Einstein manifold.*

  Then, every complete $\bar{g}$ geodesically equivalent to g

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

- **Corollary: Complete Einstein manifolds are geodesically rigid:**
  Let (M, g) is a complete pseudoriemannian Einstein manifold.

  Then, every complete $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

- **Corollary: Complete Einstein manifolds are geodesically rigid:** *Let (M, g) is a complete pseudoriemannian Einstein manifold.*
  Then, every complete $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

- If $dim(\mathcal{A}) \geq 3$, then we are done because of the following result :
  Theorem: (Kiosak, M$\sim$ April 2008) *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian manifold such that there exists a nontrivial solution of the equation (3) :*

$$f_{,ijk} = K \cdot (2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}).$$

  Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

- **Corollary: Complete Einstein manifolds are geodesically rigid:** *Let (M, g) is a complete pseudoriemannian Einstein manifold.*
  Then, every complete $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

Thanks a lot!!!