Second-order type-changing evolution equations with first-order intermediate equations

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Main Objective: Classify second-order hyperbolic-parabolic type-changing symplectic Monge-Ampère PDEs that possess an infinite set of first-order intermediate PDEs.

Traditional Definition 1 *A* Monge-Ampère PDE *is a second-order PDE of the form*

$$E\,(f_{xx}\,f_{tt} - f_{xt}^2) + A\,f_{xx} + 2B\,f_{xt} + C\,f_{tt} + D = 0, \tag{1}$$

where the coefficients are functions of the five variables (x, t, f, f_x, f_t)*.*

Unlike a general second-order PDE, a Monge-Ampère PDE can be represented by an EDS on the 5-manifold $J^1(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^5$, generated by the forms

$$\tilde{\theta} = dz - p\,dx - q\,dt$$

$$d\tilde{\theta} = -dp \wedge dx - dq \wedge dt$$

$$\begin{aligned}\widetilde{\Omega} = E\,dp \wedge dq + A\,dp \wedge dt + B\,(dq \wedge dt - dp \wedge dx) \\ +C\,dx \wedge dq + D\,dx \wedge dt,\end{aligned}$$

where the coefficients are now regarded as functions on $J^1(\mathbb{R}^2, \mathbb{R})$.

Definition 1 *A* Monge-Ampère equation *is given by a contact 5-manifold* $(\widetilde{M}, \tilde{\theta})$*, together with an effective 2-form* $\widetilde{\Omega}$ *on* $\widetilde{M}$ *(i.e.,* $\widetilde{\Omega} \wedge d\tilde{\theta} = 0$*). The corresponding* Monge-Ampère system *is the exterior differential system* $\widetilde{\mathcal{I}}$ *on* $\widetilde{M}$ *generated by* $\tilde{\theta}, d\tilde{\theta}$*, and* $\widetilde{\Omega}$*.*

In terms of local contact coordinates (x, t, z, p, q) on $\widetilde{M}$, integral manifolds of $\widetilde{\mathcal{I}}$ satisfying the independence condition $dx \wedge dt \neq 0$ are in one-to-one correspondence with solutions of (1).

If the coefficients A, B, C, D, E in (1) are independent of the variable f, then the equation can be represented by an EDS on the 4-manifold $T^*\mathbb{R}^2 \cong \mathbb{R}^4$, generated by the two 2-forms

$$\begin{aligned} \omega &= dp \wedge dx + dq \wedge dt \\ \Omega &= E\, dp \wedge dq + A\, dp \wedge dt + B\,(dq \wedge dt - dp \wedge dx) \\ &\quad + C\, dx \wedge dq + D\, dx \wedge dt. \end{aligned}$$

In this case, (1) is called a *symplectic Monge-Ampère PDE*.

Definition 2 *A* symplectic Monge-Ampère equation *is given by a symplectic 4-manifold* (M, ω)*, together with an effective 2-form* Ω *on* M*. The corresponding* symplectic Monge-Ampère system *is the exterior differential system* $\mathcal{I}$ *on* M *generated by* ω *and* Ω*.*

In terms of local symplectic coordinates (x, t, p, q) on M, integral manifolds of $\mathcal{I}$ satisfying the independence condition $dx \wedge dt \neq 0$ are in one-to-one correspondence with solutions of (1).

Note that any symplectic manifold (M, ω) with local symplectic coordinates (x, t, p, q) can be regarded as locally symplectomorphic to $T^*\mathbb{R}^2$, with local coordinates (x, t) on the base space $\mathbb{R}^2$. A choice of such a symplectomorphism gives rise to a locally defined 1-form

$$\theta = p\,dx + q\,dt$$

on M with the property that $d\theta = \omega$.

However, this choice is not canonical!

A symplectic Monge-Ampère PDE (M, ω, Ω) may (at least locally) be "partially prolonged" to a Monge-Ampère PDE $(\widetilde{M}, \tilde{\theta}, \widetilde{\Omega})$ as follows: let $\widetilde{M} = M \times \mathbb{R}$, with coordinate z on the $\mathbb{R}$ factor, and let $\rho : \widetilde{M} \to M$ be the natural projection. Define

$$\tilde{\theta} = dz - \rho^*(\theta), \qquad \widetilde{\Omega} = \rho^*(\Omega).$$

This partial prolongation satisfies the condition that

$$d\tilde{\theta} = -\rho^*(\omega).$$

Note that this construction is not canonical; it depends on the choice of local symplectomorphism $M \cong T^*\mathbb{R}^2$.

Our goal will be to classify a certain category of symplectic Monge-Ampère PDEs up to symplectomorphism.

Remark: It is possible for two symplectic Monge-Ampère PDEs to be contact equivalent after partial prolongation, but not symplectically equivalent.

Intermediate equations

While second-order PDEs are represented by effective 2-forms on (M, ω) or $(\widetilde{M}, \tilde{\theta})$, first-order PDEs are represented by functions.

Let $H : \widetilde{M} \to \mathbb{R}$ be a nondegenerate function. H determines a 1-parameter family of first-order PDEs

$$H(x, t, f, f_x, f_t) = c,$$

one for each $c \in \mathbb{R}$.

Similarly, a nondegenerate function $h : M \to \mathbb{R}$ determines a 1-parameter family of first-order PDEs

$$h(x, t, f_x, f_t) = c,$$

one for each $c \in \mathbb{R}$.

Definition 3 *An* intermediate differential equation *(IDE) for a Monge-Ampère PDE* $(\widetilde{M}, \tilde{\theta}, \widetilde{\Omega})$ *is a nondegenerate function* $H : \widetilde{M} \to \mathbb{R}$ *such that solutions for the family of first-order PDEs determined by* H *are also solutions for* $(\widetilde{M}, \tilde{\theta}, \widetilde{\Omega})$. *In terms of differential ideals, this condition may be expressed as*

$$(\tilde{\theta}, d\tilde{\theta}, \widetilde{\Omega}) \subset (\tilde{\theta}, d\tilde{\theta}, dH),$$

or, equivalently,

$$\widetilde{\Omega} \in (\tilde{\theta}, d\tilde{\theta}, dH). \tag{2}$$

Classically, an IDE for a symplectic Monge-Ampère PDE (M, ω, Ω) is simply an IDE for any partial prolongation. But because the partial prolongation construction is non-canonical, it would be preferable to define IDEs for (M, ω, Ω) in terms of objects on M, independent of any particular partial prolongation.

In order to accomplish this, we need to distinguish between two types of IDES:

Provisional Definition 1 *Given a symplectic Monge-Ampère PDE* (M, ω, Ω)*, a partial prolongation* $\rho : \widetilde{M} \to M$*, and an IDE* $H : \widetilde{M} \to \mathbb{R}$*, we say that* H *is*

- cylindrical *if the level sets* $H = c$ *are ruled by the* $\mathbb{R}$*-fibers of* ρ *(i.e., if* $H = \rho^*(h)$ *for some function* $h : M \to \mathbb{R}$*);*
- graph-like *if the level sets* $H = c$ *are transverse to the* $\mathbb{R}$*-fibers of* ρ*.*

Remark: The term "graph-like" is motivated by the fact that, by the implicit function theorem, any first-order PDE $H = c$ in the family determined by a graph-like IDE $H : \widetilde{M} \to \mathbb{R}$ can locally be written in the form $z = \rho^*(h_c)$ for some function $h_c : M \to \mathbb{R}$.

First, consider a CIE $H = \rho^*(h),\ h : M \to \mathbb{R}$. Condition (3) is equivalent to

$$\rho^*(\Omega) \in (\tilde{\theta}, \rho^*(\omega), \rho^*(dh)),$$

which in turn holds if and only if

$$\Omega \in (\omega, dh).$$

In terms of ideals, we write this condition as

$$(\omega, \Omega) \subset (\omega, dh). \tag{3}$$

Key point: Condition (3) is independent of the choice of partial prolongation $\rho : \widetilde{M} \to M$.

Next, consider the GIEs. We make use of the following lemma:

Lemma 1 *The collection of GIEs for a symplectic Monge-Ampère PDE is generated by functions of the form* $H = z - h(x, t, p, q)$.

With H as in the lemma, (3) is equivalent to

$$\begin{aligned} \rho^*(\Omega) &\in (\tilde{\theta}, \rho^*(\omega), dz - \rho^*(dh)) \\ &= (\tilde{\theta}, \rho^*(\omega), \rho^*(\theta - dh)). \end{aligned}$$

(Recall that in local symplectic coordinates, $\theta = p\,dx + q\,dt$.)

This condition holds if and only if

$$\Omega \in (\omega, \theta - dh).$$

In terms of ideals, we write this condition as

$$(\omega, \Omega) \subset (\omega, \theta - dh). \tag{4}$$

Key point: Condition (4) is independent of the choice of partial prolongation $\rho : \widetilde{M} \to M$.

Notation: Let θ_h denote the 1-form $\theta - dh$ on M.

For both CIEs and GIEs, the important object is not the function h, but rather a 1-form: dh in the case of a CIE or θ_h in the case of a GIE. These 1-forms may locally be distinguished by the conditions that

- $d(dh) = 0$ – i.e., dh is exact;
- $d(\theta_h) = \omega$.

We are now ready to define CIEs and GIEs as objects on M:

Definition 4 *Let* (M, ω, Ω) *be a symplectic Monge-Ampère PDE. An* intermediate differential equation (IDE) *for* (M, ω, Ω) *is a* 1*-form* α *on* M *satisfying the conditions that:*

- $d\alpha = \lambda\omega, \qquad \lambda \in \{0, 1\}$,
- $(\omega, \Omega) \subset (\omega, \alpha)$.

If $\lambda = 0$, *we say that* α *is a* cylindrical intermediate equation (CIE)*; if* $\lambda = 1$, *we say that* α *is a* graph-like intermediate equation (GIE)*.*

The symplectic characteristic variety

Definition 5 *For any point $e \in M$, the* symplectic characteristic variety SCV_e *of* (M, ω, Ω) *at* e *is the cone*

$$\{\mathbf{v} \in T_e M \mid (\mathbf{v} \lrcorner \omega) \wedge (\mathbf{v} \lrcorner \Omega) = 0\}.$$

SCV_e consists of precisely those vectors in $T_e M$ for which the extension to an integral element is *not* unique.

We will generally find it more convenient to work with the "symplectically dual" object

$$\mathrm{SCV}_e^* = \{\mathbf{v} \lrcorner \omega \mid \mathbf{v} \in \mathrm{SCV}_e\}.$$

We will use the term "symplectic characteristic variety" to refer to either SCV_e or SCV_e^*.

Proposition 1 *Let α be a nowhere-zero 1-form on M satisfying*

$$d\alpha = \lambda\,\omega, \ \ \lambda \in \{0, 1\}.$$

Then α is an IDE for the symplectic Monge-Ampère PDE (M, ω, Ω) if and only if $\alpha \in \mathrm{SCV}^*$.

Proof: Let X_α be the unique vector field on M satisfying

$$X_\alpha \lrcorner \omega = -\alpha.$$

Note that $\alpha \in \mathrm{SCV}^*$ if and only if $X_\alpha \in \mathrm{SCV}$.

($\Rightarrow$): Suppose that α is an IDE for (M, ω, Ω). Since $d\alpha = \lambda\omega$ with $\lambda \in \{0, 1\}$, the system (ω, α) is differentially closed. Furthermore, X_α is a Cauchy characteristic vector field for this system; i.e.,

$$X_\alpha \lrcorner (\omega, \alpha) \subset (\omega, \alpha).$$

By definition, $(\omega, \Omega) \subset (\omega, \alpha)$; therefore, $X_\alpha \lrcorner \Omega \in (\omega, \alpha)$. Since $X_\alpha \lrcorner \Omega$ is a 1-form, it follows that it must be a multiple of $\alpha = -X_\alpha \lrcorner \omega$. Therefore,

$$(X_\alpha \lrcorner \omega) \wedge (X_\alpha \lrcorner \Omega) = 0,$$

and $X_\alpha \in \mathrm{SCV}$, as desired.

($\Leftarrow$): Suppose that $\alpha \in \mathrm{SCV}^*$. Then $X_\alpha \in \mathrm{SCV}$, and so

$$(X_\alpha \lrcorner \omega) \wedge (X_\alpha \lrcorner \Omega) = 0.$$

Therefore,

$$(X_\alpha \lrcorner \Omega) = \mu (X_\alpha \lrcorner \omega) = -\mu\, \alpha$$

for some function μ on M.

Now consider the 2-form $\Omega + \mu\, \omega$, and compute its wedge product with α (recalling that $\Omega \wedge \omega = 0$):

$$\begin{aligned}
\alpha \wedge (\Omega + \mu\, \omega) &= -(X_\alpha \lrcorner \omega) \wedge (\Omega + \mu\, \omega) \\
&= -X_\alpha \lrcorner [\omega \wedge (\Omega + \mu\, \omega)] + \omega \wedge [X_\alpha \lrcorner (\Omega + \mu\, \omega)] \\
&= -X_\alpha \lrcorner (\mu\, \omega \wedge \omega) - \omega \wedge (2\mu\, \alpha) \\
&= \mu(\alpha \wedge \omega + \omega \wedge \alpha) - 2\mu\, \omega \wedge \alpha \\
&= 0.
\end{aligned}$$

Therefore,

$$\Omega + \mu\,\omega \equiv 0 \mod (\alpha)$$
$$\Rightarrow\ \Omega \equiv 0 \mod (\omega,\, \alpha).$$

It follows that

$$(\omega, \Omega) \subset (\omega, \alpha),$$

and α is an IDE for (M, ω, Ω), as desired. □

Elliptic, hyperbolic, and parabolic types

For any symplectic Monge-Ampère PDE (M, ω, Ω), we can write

$$\Omega \wedge \Omega = \tau \, \omega \wedge \omega$$

for some real-valued function $\tau : M \to \mathbb{R}$, which is well-defined up to multiplication by a positive function. A point $e \in M$ is called:

- an *elliptic point* if $\tau(e) > 0$;
- a *hyperbolic point* if $\tau(e) < 0$;
- a *parabolic point* if $\tau(e) = 0$ and $\Omega_e \neq 0$;
- a *zero point* if $\Omega_e = 0$;
- a *parabolic point of type change* if e is a parabolic point and τ is not identically zero on any neighborhood of e;
- *a zero point of type change* if e is a zero point and τ is not identically zero on any neighborhood of e.

A local coframing $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ on M is called *symplectic* if

$$\omega = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4.$$

It is well-known that:

- If e is an elliptic point, then $\mathrm{SCV}_e = \{0\}$, and there exists a local symplectic coframing such that

$$\Omega_e = \omega_2 \wedge \omega_3 + \omega_1 \wedge \omega_4.$$

- If e is a hyperbolic point, then SCV_e consists of a pair of 2-planes V_e, $\overline{V}_e$ satisfying $V_e \cap \overline{V}_e = \{0\}$ and $\omega(V_e, \overline{V}_e) = 0$. Furthermore, there exists a local symplectic coframing such that

$$\Omega_e = \omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_4.$$

In terms of this coframing, we have

$$V_e = \{\omega_1, \omega_2\}^\perp, \qquad \overline{V}_e = \{\omega_3, \omega_4\}^\perp.$$

Every integral element in $T_e M$ intersects SCV_e in two lines, one lying in V_e and the other in $\overline{V}_e$. In this case,

$$\mathrm{SCV}_e^* = V_e^\perp \cup \overline{V}_e^\perp = \{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4\}.$$

- If e is a parabolic point, then SCV_e consists of a single 2-plane on which ω vanishes. Furthermore, there exists a local symplectic coframing such that

$$\Omega_e = \omega_2 \wedge \omega_3.$$

In terms of this coframing, we have

$$\mathrm{SCV}_e = \{\omega_2, \omega_3\}^\perp.$$

Every integral element in $T_e M$ either intersects SCV_e in a line or coincides with SCV_e. In this case,

$$\mathrm{SCV}_e^* = V_e^\perp = \{\omega_2, \omega_3\}.$$

- If e is a zero point, then $\mathrm{SCV}_e = T_e M$.

Normal forms for hyperbolic PDEs with IDEs

Theorem 1 *If* (M, ω, Ω) *is hyperbolic and* $V^{\perp}$, $\overline{V}^{\perp}$ *each contain a* 1*-dimensional integrable subsystem, then at any point* $e \in M$, (M, ω, Ω) *is locally symplectically equivalent to*

$$\Omega = B\,(dq \wedge dt - dp \wedge dx) + C\,dx \wedge dq, \tag{5}$$

where the point e *corresponds to* $(x, t, p, q) = (0, 0, 1, 1)$ *and* $B \neq 0$*. This equation has CIEs of the form* $d(h(q))$, $d(\overline{h}(x))$, *where* h, $\overline{h}$ *are arbitrary nondegenerate* $\mathcal{C}^{\infty}$*-functions of one variable.*

Remarks:

- The 2-form (5) corresponds to the evolution equation

$$2B(x,t,f_x,f_t)\,f_{xt} + C(x,t,f_x,f_t)\,f_{tt} = 0.$$

- The choice of local coordinates satisfying $(x,t,p,q) = (0,0,1,1)$ at e is in keeping with with the local normal forms that we will construct near parabolic and zero points later.

Proof: We can choose a local symplectic coframing satisfying

$$V^{\perp} = \{\omega_1, \omega_2\}, \qquad \overline{V}^{\perp} = \{\omega_3, \omega_4\},$$

and this coframing may be chosen so that ω_1 spans an integrable subsystem of $V^{\perp}$ and ω_3 spans an integrable subsystem of $\overline{V}^{\perp}$. Multiplying by nonvanishing functions, we can arrange that $\omega_1 = dq$ and $\omega_3 = dx$ for some independent functions x and q.

Furthermore,

$$dx \wedge dq \wedge \omega = 0.$$

It follows from a theorem of Liouville that we can complete (x, q) to a local symplectic coordinate system (x, t, p, q), so that

$$\omega = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 = dq \wedge dt + dp \wedge dx. \tag{6}$$

Now $\omega \wedge dx = dq \wedge dt \wedge dx = dq \wedge \omega_2 \wedge dx$; therefore,

$$\omega_2 \equiv dt + \gamma\, dx \pmod{dq}.$$

Similarly, $\omega \wedge dq = dp \wedge dx \wedge dq = dx \wedge \omega_4 \wedge dq$; therefore,

$$\omega_4 \equiv -dp + \delta\, dq \pmod{dx}.$$

Next, equation (6) implies that $\delta = \gamma$. If we set $Q = \frac{C}{2B} = \delta = \gamma$ (with $B \neq 0$), then since Ω is only determined up to a nonzero multiple, we can write

$$\begin{aligned} \omega &= dq \wedge (dt - Q\, dx) + (dp + Q\, dq) \wedge dx \\ \Omega &= B\Big(dq \wedge (dt - Q\, dx) - (dp + Q\, dq) \wedge dx\Big) \\ &= B(dq \wedge dt - dp \wedge dx) + C\, dx \wedge dq, \end{aligned} \tag{7}$$

as claimed. □

Observe that, for any nondegenerate $\mathcal{C}^\infty$-functions $h, \bar{h}$ of one variable, we have

$$d(h(q)) = h'(q)\, dq \in V^\perp, \qquad d(\bar{h}(x)) = \bar{h}'(x)\, dx \in \overline{V}^\perp;$$

therefore, by Proposition 1, $d(h(q))$ and $d(\bar{h}(x))$ are CIEs for (5). □

Corollary 1 *Under the conditions of Theorem 1, the components of* SCV* *take the form*

$$V^\perp = \{dq,\, dt - Q\, dx\}, \qquad \overline{V}^\perp = \{dx,\, dp + Q\, dq\}.$$

If (5) possesses additional IDEs, then this normal form admits several possible refinements, depending on whether the additional IDEs are GIEs or CIEs.

Theorem 2 *Let* (M, ω, Ω) *be as in Theorem 1, and suppose that* $V^{\perp}$, $\overline{V}^{\perp}$ *each contain at least one GIE. Then at any point* $e \in M$, (M, ω, Ω) *is locally symplectically equivalent to* (5), *where* e *corresponds to* $(0, 0, 1, 1)$ *and* $Q_{pt} = a$ *for an invariant real constant* a. *Specifically:*

- *If* $a \neq 0$, *then* (M, ω, Ω) *has local normal form* (5) *with*

$$Q = \frac{C}{2B} = ap(t+1),$$

corresponding to the PDE

$$f_{xt} + a(t+1) f_x f_{tt} = 0.$$

The GIEs belonging to $V^{\perp}$, $\overline{V}^{\perp}$ *are represented by the 1-forms* $\theta_h, \theta_{\bar{h}}$, *where*

$$h = t\,q + \frac{1}{a}\ln(t+1) + k(q)$$
$$\bar{h} = (t+1)\,q + \frac{1}{a}\ln|p| + \bar{k}(x),$$

and k *and* $\bar{k}$ *are arbitrary* $\mathcal{C}^{\infty}$*-functions of one variable.*

- *If* $a = 0$, *then* (M, ω, Ω) *has local normal form* (5) *with*

$$Q = \frac{C}{2B} = \frac{p - t}{x + q + 1},$$

corresponding to the PDE

$$f_{xt} + \left(\frac{f_x - t}{x + f_t + 1} \right) f_{tt} = 0.$$

The GIEs belonging to $V^{\perp}$, $\overline{V}^{\perp}$ *are represented by the 1-forms* $\theta_h, \theta_{\bar{h}}$, *where*

$$\begin{aligned} h &= t\,q + t(x + q + 1) + k(q) \\ \bar{h} &= t\,q - p(x + q + 1) + \bar{k}(x), \end{aligned}$$

and k *and* $\bar{k}$ *are arbitrary* $\mathcal{C}^{\infty}$*-functions of one variable.*

Theorem 3 *Let* (M, ω, Ω) *be as in Theorem 1, and suppose that* $V^\perp$ *contains at least one GIE and* $\overline{V}^\perp$ *is completely integrable. Then at any point* $e \in M$, (M, ω, Ω) *is locally symplectically equivalent to* (5)*, where* e *corresponds to* $(0, 0, 1, 1)$ *and* $Q = \frac{C}{2B} = \frac{p}{2}$*, corresponding to the PDE*

$$f_{xt} + \tfrac{1}{2} f_x f_{tt} = 0.$$

The CIEs belonging to $\overline{V}^\perp$ *are represented by the 1-form* $d\bar{h}$*, where*

$$\bar{h} = \bar{h}(x, p^2 e^q),$$

and $\bar{h}$ *is an arbitrary* $\mathcal{C}^\infty$*-function of two variables. The GIEs belonging to* $V^\perp$ *are represented by the 1-form* θ_h*, where*

$$h = t(q + 2) + k(q),$$

and k *is an arbitrary* $\mathcal{C}^\infty$*-function of one variable.*

Theorem 4 *Let (M, ω, Ω) be as in Theorem 1, and suppose that $V^{\perp}$, $\overline{V}^{\perp}$ are both completely integrable. Then at any point $e \in M$, (M, ω, Ω) is locally symplectically equivalent to (5), where e corresponds to $(0, 0, 1, 1)$ and $Q = 1$, corresponding to the PDE*

$$f_{xt} + f_{tt} = 0.$$

The CIEs are given by $dh, d\bar{h}$, where

$$h = h(q, t - x), \qquad \bar{h} = \bar{h}(x, p + q),$$

and h, $\bar{h}$ are arbitrary $\mathcal{C}^{\infty}$-functions of 2-variables.

This theorem is, in fact, the classical prototype for all our theorems; it was known to Lie and Darboux. The normal form in this theorem is equivalent to the classical wave equation

$$f_{xt} = 0.$$

Normal forms for type-changing PDEs with IDEs

Now suppose that (M, ω, Ω) is hyperbolic on a dense open subset $\mathcal{H}$ of M (called the *hyperbolic locus*), and that $M \setminus \mathcal{H}$ consists of a set $\mathcal{P}$ of parabolic points of type change (called the *parabolic locus*) and a (possibly empty) set $\mathcal{Z}$ of zero points (called the *zero locus*). If

$$\Omega = B\,(dq \wedge dt - dp \wedge dx) + C\,dx \wedge dq,$$

then

$$\mathcal{P} = \{(x,t,p,q) \in M \mid B(x,t,p,q) = 0 \text{ and } C(x,t,p,q) \neq 0\},$$

and

$$\mathcal{Z} = \{(x,t,p,q) \in M \mid B(x,t,p,q) = C(x,t,p,q) = 0\}.$$

In general, SCV* need not extend smoothly to the parabolic locus. We will use IDEs to locally characterize a class of evolution equations for which both components $V^\perp$, $\overline{V}^\perp$ extend smoothly from the hyperbolic locus $\mathcal{H}$ to the parabolic locus $\mathcal{P}$. Specifically, our key involutivity assumption will be that the type-changing locus $\mathcal{P} \cup \mathcal{Z}$ is defined by a CIE. For this class, $\mathcal{P}$ is a regular hypersurface, and parabolicity implies that $V^\perp$ coincides with $\overline{V}^\perp$ at each point of $\mathcal{P}$.

If the PDE has a nonempty zero locus $\mathcal{Z} \subset M$, then $V^\perp$ and $\overline{V}^\perp$ will *not* extend smoothly to $\mathcal{Z}$; specifically, SCV$^* = \{0\}$ at each point of $\mathcal{Z}$.

Definition 6 *Let (M, ω, Ω) be a symplectic Monge-Ampère PDE for which every point of M is either a hyperbolic point, a parabolic point of type change, or a zero point. Suppose that the parabolic locus $\mathcal{P} \subset M$ is 3-dimensional, and that the zero locus $\mathcal{Z} \subset M$ is a 2-dimensional Lagrangian surface. We say that (M, ω, Ω) is* involutive type-changing of order m, *$m \in \mathbb{Z}_+$, if:*

- *the closure of $\mathcal{P}$ is equal to $\mathcal{P} \cup \mathcal{Z}$;*
- *there exists a $\mathcal{C}^\infty$-function $q : M \to \mathbb{R}$, with dq nowhere-vanishing, such that the zero locus of q is $\mathcal{P} \cup \mathcal{Z}$;*
- *the components $V^\perp, \overline{V}^\perp$ of* SCV* *each extend smoothly (as rank 2 Pfaffian systems) to the parabolic locus $\mathcal{P}$;*
- *dq is a CIE belonging to $V^\perp$ at each point of $M - \mathcal{Z}$;*
- *$\Omega \wedge \Omega = \tau\, \omega \wedge \omega$, where $\tau = q^{2m} \hat{\tau}$, and $\hat{\tau}$ is $\mathcal{C}^\infty$ and nonvanishing.*

We will say that (M, ω, Ω) *is* involutive type-changing of order m with CIEs *if, in addition, there exists another function* $x : M \to \mathbb{R}$ *such that:*

- dx *is a CIE belonging to* $\overline{V}^{\perp}$ *at each point of* $M - \mathcal{Z}$*;*
- *the pullback of* dx *to* $\mathcal{Z}$ *is a nonvanishing 1-form on* $\mathcal{Z}$*;*
- $dq \wedge dx$ *is a nonvanishing* 2*-form on* M*.*

We will prove analogs of the preceding theorems for hyperbolic PDEs with IDEs for involutive type-changing PDEs with CIEs. The following lemma will be useful:

Lemma 2 *Let* (M, ω, Ω) *be involutive type-changing of order* m *with CIEs.*

1. *In a neighborhood of any parabolic point* $e \in M$*, we can complete* (x, q) *to a symplectic coordinate system* (x, t, p, q) *(i.e., a coordinate system for which* $\omega = dq \wedge dt + dp \wedge dx$*) such that* e *corresponds to the point* $(x, t, p, q) = (0, 0, 1, 0)$*.*
2. *In a neighborhood of any zero point* $e \in M$*, we can complete* (x, q) *to a symplectic coordinate system* (x, t, p, q) *such that* e *corresponds to the point* $(x, t, p, q) = (0, 0, 0, 0)$*, and the zero locus near* e *has the form* $\mathcal{Z} = \{q = p = 0\}$*.*

Proof: Because dx and dq lie in distinct characteristic subsystems, (1) is immediate. For case (2), let (x, t, p, q) be any completion of (x, q) to local symplectic coordinates in a neighborhood of e such that e corresponds to $(0, 0, 0, 0)$. Since $\mathcal{Z}$ is a Lagrangian surface on which $q = 0$ and dx is nonvanishing, it must have the form

$$\mathcal{Z} = \{q = p - \varphi(x) = 0\}$$

for some function $\varphi(x)$ with $\varphi(0) = 0$. By making the symplectic coordinate transformation

$$(x, t, p, q) \mapsto (x, t, p + \varphi(x), q)$$

we can assume that $\mathcal{Z}$ has the form

$$\mathcal{Z} = \{q = p = 0\},$$

as desired.

We have the following analog of Theorem 1:

Theorem 5 *If (M, ω, Ω) is involutive type-changing of order m with CIEs, then at any point $e \in \mathcal{P}$, (M, ω, Ω) is locally symplectically equivalent to the normal form*

$$\Omega = q^m\,(dq \wedge dt - dp \wedge dx) + C\,dx \wedge dq, \tag{8}$$

where the point e corresponds to $(x, t, p, q) = (0, 0, 1, 0)$, and C is a smooth function which is nonzero on the parabolic locus $\mathcal{P}$. Furthermore, at any point $e \in \mathcal{Z}$ a similar conclusion holds, where the point e now corresponds to $(x, t, p, q) = (0, 0, 0, 0)$, with the additional condition that C is a $\mathcal{C}^\infty$-function which vanishes on the zero locus $\mathcal{Z} = \{q = p = 0\}$.

Remark: The 2-form (8) corresponds to the evolution equation

$$2 f_t^m f_{xt} + C(x, t, f_x, f_t) f_{tt} = 0.$$

Proof: Choose local symplectic coordinates (x, t, p, q) in a neighborhood of e as in Lemma 2. We can divide Ω by $\sqrt{|\hat{\tau}|}$ to arrange that $\tau = -q^{2m}$. Near this point, the characteristic systems are given by

$$V^{\perp} = \{dq,\ q^m\, dt - \tfrac{1}{2} C\, dx\}, \qquad \overline{V}^{\perp} = \{dx,\ q^m\, dp + \tfrac{1}{2} C\, dq\},$$

and we can write

$$\Omega = q^m \,(dq \wedge dt - dp \wedge dx) + C\, dx \wedge dq$$

for some $\mathcal{C}^{\infty}$ function C.

If $e \in \mathcal{P}$, then C must be nonzero on a neighborhood of e. If $e \in \mathcal{Z}$, then C must vanish precisely on the intersection of the local symplectic coordinate neighborhood with $\mathcal{Z}$.

We also have the following analog of Corollary 1:

Corollary 2 *Under the conditions of Theorem 5, the components of* SCV* *take the form*

$$V^\perp = \{dq,\, q^m dt - \tfrac{1}{2}C\,dx\}, \qquad \overline{V}^\perp = \{dx,\, q^m dp + \tfrac{1}{2}C\,dq\}.$$

We now restrict our attention to the case $m = 1$ in order to keep the normal forms manageable.

We have the following analog of Theorem 2:

Theorem 6 *Let (M, ω, Ω) be involutive type-changing of order 1 with CIEs, and suppose that $V^{\perp}$, $\overline{V}^{\perp}$ each contain at least one GIE. Then at any point e in $\mathcal{P}$ or $\mathcal{Z}$, (M, ω, Ω) is locally symplectically equivalent to (8), where e corresponds to $(0, 0, 1, 0)$ if $e \in \mathcal{P}$ and to $(0, 0, 0, 0)$ if $e \in \mathcal{Z}$, and $C_{pt} = 2aq$ for an invariant real constant a. Specifically:*

- *If* $a \neq 0$, *then* (M, ω, Ω) *has local normal form* (8) *with*

$$C = 2ap(tq + b)$$

for an invariant nonzero constant b, *corresponding to the PDE*

$$f_t f_{xt} + a f_x (t f_t + b) f_{tt} = 0.$$

The GIEs belonging to $V^\perp$, $\overline{V}^\perp$ *are represented by the 1-forms* $\theta_h, \theta_{\bar{h}}$, *where*

$$h = t\,q + \frac{1}{a} \ln |tq + b| + k(q)$$
$$\bar{h} = tq + b \ln |q| + \frac{1}{a} \ln |p| + \bar{k}(x),$$

and k *and* $\bar{k}$ *are arbitrary* $\mathcal{C}^\infty$*-functions of one variable.*

- *If* $a = 0$*, then* $\mathcal{Z}$ *is empty and* (M, ω, Ω) *has local normal form* (8) *with*

$$C = \frac{2q\,(\phi'(q)p - t)}{\phi(q) + x + 1} + c_0(x)$$

for some nonvanishing $\mathcal{C}^\infty$*function* $c_0(x)$ *and some* $\mathcal{C}^\infty$*function* $\phi(q)$ *satisfying* $\phi(0) = 0$ *and* $\phi'(q) \neq 0$ *when* $q \neq 0$*. The corresponding PDE is*

$$f_t f_{xt} + \left(\frac{f_t(\phi'(f_t)f_x - t)}{\phi(f_t) + x + 1} + c_0(x) \right) f_{tt} = 0.$$

The GIEs belonging to $V^\perp$, $\overline{V}^\perp$ are represented by the 1-forms $\theta_h, \theta_{\bar{h}}$, where

$$\begin{aligned} h &= t\,q + k(q) \\ &\qquad + \frac{1}{\phi'(q)}\left(t(\phi(q)+x+1) - \frac{1}{2q}\int c_0(x)(\phi(q)+x+1)\,dx\right) \\ \bar{h} &= t\,q - p(\phi(q)+x+1) + \bar{k}(x) \\ &\qquad - \frac{1}{2}c_0(x)\left((x+1)\ln|q| + \int \frac{\phi(q)}{q}\,dq\right), \end{aligned}$$

and k and $\bar{k}$ are arbitrary $\mathcal{C}^\infty$-functions of one variable.

Remark: Many of these GIEs do not extend smoothly to the parabolic locus.

Theorem 3 has two distinct analogs because in the involutive type-changing case, $V^\perp$ is singled out as being the characteristic subsystem containing the CIE that defines the type-changing locus. The two versions depend on whether $V^\perp$ or $\overline{V}^\perp$ is the completely integrable subsystem.

Theorem 7 *Let* (M, ω, Ω) *be involutive type-changing of order* 1 *with CIEs, and suppose that* $V^{\perp}$ *contains at least one GIE and* $\overline{V}^{\perp}$ *is completely integrable. Then at any point* e *in* $\mathcal{P}$ *or* $\mathcal{Z}$*,* (M, ω, Ω) *is locally symplectically equivalent to* (8)*, where* e *corresponds to* $(0, 0, 1, 0)$ *if* $e \in \mathcal{P}$ *and to* $(0, 0, 0, 0)$ *if* $e \in \mathcal{Z}$*, and* $C_p = 2b$ *for an invariant real, nonzero constant* b*. Specifically:*

- *If* b *is not a negative integer, then* (M, ω, Ω) *has local normal form (8) with*

$$C = 2bp,$$

corresponding to the PDE

$$f_t f_{xt} + b f_x f_{tt} = 0.$$

The CIEs belonging to $\overline{V}^{\perp}$ *are represented by the* 1*-form* $d\bar{h}$*, where*

$$\bar{h} = \bar{h}(x, pq^b)$$

and $\bar{h}$ *is an arbitrary* $\mathcal{C}^{\infty}$ *function of two variables. The GIEs belonging to* $V^{\perp}$ *are represented by the 1-form* θ_h*, where*

$$h = \left(\frac{b+1}{b}\right) tq + k(q)$$

and k *is an arbitrary* $\mathcal{C}^{\infty}$ *function of one variable.*

- *If b is a negative integer, then (M, ω, Ω) has either the local normal form above, or the local normal form (8) with*

$$C = 2bp + 2c(x)q^{-b}$$

for some $\mathcal{C}^\infty$ function $c(x)$, corresponding to the PDE

$$f_t f_{xt} + (bf_x + c(x)(f_t)^{-b})f_{tt} = 0.$$

In the latter case, the CIEs belonging to $\overline{V}^\perp$ are represented by the 1-form $d\bar{h}$, where

$$\bar{h} = \bar{h}(x,\, pq^b + c(x)\ln|q|)$$

and $\bar{h}$ is an arbitrary $\mathcal{C}^\infty$ function of two variables, and the GIEs belonging to $V^\perp$ are represented by the 1-form θ_h, where

$$h = \left(\frac{b+1}{b}\right) tq - \left(\frac{1}{b}\right) q^{-b} \int c(x)\, dx + k(q)$$

and k is an arbitrary $\mathcal{C}^\infty$ function of one variable.

Theorem 8 *Let* (M, ω, Ω) *be involutive type-changing of order* 1 *with CIEs, and suppose that* $V^\perp$ *is completely integrable and* $\overline{V}^\perp$ *contains at least one GIE. Then* $\mathcal{Z}$ *is empty, and at any point* $e \in \mathcal{P}$, (M, ω, Ω) *is locally symplectically equivalent to* (8), *where* e *corresponds to* $(0, 0, 1, 0)$ *and*

$$C = 2tq + 2c_0(x)$$

for some nonvanishing function $c_0(x)$, *corresponding to the PDE*

$$f_t f_{xt} + (t f_t + c_0(x)) f_{tt} = 0.$$

The CIEs belonging to $\overline{V}$ *are represented by the* 1*-form* dh*, where*

$$h = h\big(q,\, 2tqe^{-x} - \int c_0(x)e^{-x}\,dx\big)$$

and h *is an arbitrary* $\mathcal{C}^\infty$*function of two variables. The GIEs belonging to* $V^\perp$ *are represented by the 1-form* $\theta_{\bar{h}}$*, where*

$$\bar{h} = tq + p + \tfrac{1}{2}c_0(x)\ln|q| + \bar{k}(x)$$

and $\bar{k}$ *is an arbitrary* $\mathcal{C}^\infty$*function of one variable.*

We have the following analog of Theorem 4:

Theorem 9 *Let* (M, ω, Ω) *be involutive type-changing of order* 1 *with CIEs, and suppose that* $V^{\perp}$, $\overline{V}^{\perp}$ *are both completely integrable. Then* $\mathcal{Z}$ *is empty, and at any point* $e \in \mathcal{P}$, (M, ω, Ω) *is locally symplectically equivalent to* (8), *where* e *corresponds to* $(0, 0, 1, 0)$ *and* $C = 2$, *corresponding to the PDE*

$$f_t f_{xt} + f_{tt} = 0.$$

The CIEs belonging to $V^{\perp}$ *and* $\overline{V}^{\perp}$ *are represented by the* 1-*forms* dh *and* $d\bar{h}$, *respectively, where*

$$h = h(q, tq - x), \qquad \bar{h} = \bar{h}(x, qe^{p}),$$

and h, $\bar{h}$ *are arbitrary* $\mathcal{C}^{\infty}$ *functions of two variables.*

Main example: In 1998, P. Michor and T. Ratiu defined a natural energy functional on the set of curves in the space $C^\infty(\mathbb{R}, \mathbb{E})$ of smooth functions $f : \mathbb{R} \to \mathbb{E}$. A function

$$f : \mathbb{R} \times [0, T] \to \mathbb{E}$$

can be thought of as a curve $\gamma_f : [0, T] \to C^\infty(\mathbb{R}, \mathbb{E})$, where for any fixed $t \in [0, T]$,

$$\gamma_f(t) = \Big(x \to f(x, t)\Big) \in C^\infty(\mathbb{R}, \mathbb{E}).$$

The natural "tangent vector" to such a curve γ_f is the vector field

$$V(x,t) = f_t(x,t)\frac{d}{dz}\bigg|_{z=f(x,t)}$$

on $\mathbb{E}$, so it is natural to define the energy functional

$$\begin{aligned}\mathcal{E}(\gamma_f) &= \int_0^T \tfrac{1}{2}\langle V, V\rangle\,dt \\ &= \tfrac{1}{2}\int_0^T \int_{\mathbb{E}} f_t^2 dz\,dt \\ &= \tfrac{1}{2}\int_0^T \int_{\mathbb{R}} f_t^2 f_x\,dx\,dt. \end{aligned} \tag{9}$$

The Euler-Lagrange equation for this functional is

$$2f_t f_{xt} + f_x f_{tt} = 0. \tag{10}$$

This is an equation of the type in Theorem 7; it has CIEs of the form

$$f_x^2 f_t = r(x)$$

and GIEs of the form

$$f = 3t f_t + k(f_t).$$

Michor and Ratiu constructed global solutions for this equation in the case where the initial function $f(x, 0) : \mathbb{R} \to \mathbb{E}$ is an orientation-preserving embedding—i.e., when the function $p_0(x) = f_x(x, 0)$ is strictly positive. Equation (10) is well-posed for such initial data.

If initial data

$$\bar{p}_0(x) = f_x(x, 0), \qquad \bar{q}_0(x) = f_t(x, 0)$$

for (10) has the property that $\bar{p}_0(0) = 0$ and $\bar{q}_0(0) \neq 0$, then the corresponding initial value problem is ill-posed in the classical sense; i.e., $f_{tt}(0)$ cannot be expressed in terms of the initial conditions.

However, this initial data yields well-posed initial data for the GIEs. In effect, the GIEs can be used to "repair" the original ill-posed initial value problem.

On the other hand, if $\bar{q}_0(0) = \bar{p}_0(0) = 0$, then the initial conditions intersect the zero locus $\mathcal{Z}$, and there is no local existence theory for the second-order initial value problem.

Nevertheless, we have been able to exploit the CIEs of this equation to construct global solutions for such initial data, and even more general initial data where the graph of $f_0(x)$ contains cusps.

The CIEs are still ill-posed for this type of initial data, but by using careful geometric arguments together with the method of characteristics, we are able to choose appropriate functions $r(x)$ and initial data $\bar{p}_0(x)$ in order to guarantee the existence of global solutions with certain types of singularities.