The transition from arithmetic to algebra

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H. Wu

What can't our students achieve algebra?

One reason is that we do not properly prepare them for it. We teach them the wrong things.

Or, in less polite terms,

garbage in, garbage out.

We get the failure that we richly deserve.

Take the case of students' inability to do word problems in algebra. In the early grades, it is common for teachers to teach students to watch for **key words**:

> "and", "more", etc., call for $+$,
>
> "of", "as much", etc., call for $\times$,

and so on.

While this works in K–2 or even grade 3, it won't work well in grade 5 and up. But once the habit of looking for key words is formed, students cease to look for meaning because their encounters with word problems only provoke the conditioned reflex of **instant alienation** from the words.

Under the circumstance, a standard problem such as

> Working at a constant rate, two workmen can paint a house together in 6 days. In how many days can each paint it alone if it takes one of them 5 days longer than the other?

would be totally incomprehensible. Students who were brought up with this conditioned reflex do not know where to begin on such a problem.

I will have more to say at the end.

In the remainder of this presentation, I would like to concentrate on the teaching of fractions and its impact on learning algebra.

A main difference between arithmetic and algebra is that arithmetic is essentially about computations with specific numbers ($121 - 89 =?$, $\dfrac{11/4}{5/21} \times 15 =?$), whereas algebra introduces the concept of generality and abstraction.

Generality and abstraction go hand-in-hand. The two things cannot be separated.

In algebra, the computation is generally not with any specific number, but with numbers that belong to an infinite collection (all integers, all positive fractions, etc.), or with a fixed number that belongs to a given infinite collection but is otherwise not yet determined, e.g., for a number x,

$$\frac{\frac{6}{5}x^2-4}{2x-3}+\frac{3}{x^4+1}=\frac{(\frac{6}{5}x^2-4)(x^4+1)+3\,(2x-3)}{(2x-3)(x^4+1)}$$

Conceptually, this is no different from

$$\frac{2}{3}+\frac{7}{4}=\frac{(2\times 4)+(7\times 3)}{3\times 4},$$

but the former requires the ability to see each of $\frac{6}{5}x^2-4$, $2x-3$, x^4+1 as a single number for *any* number x.

This is a simple illustration of abstract thinking.

Another example: what number x would satisfy

$$x^2 + x - \frac{1}{2} = 0 \text{ ?}$$

Here one has to compute with the number x, *without* knowing what x is.

Abstract thinking again.

Incidentally, the only way one can compute with $x^2 + x - \frac{1}{2}$ is by use of the associative laws, commutative laws, and the distributive law for **all** numbers. How likely is it that we impress on our pre-algebra students that this is why they must come to terms with these (to them) *utterly boring* laws?

The natural progression in the curriculum of grades 3–8 is:

whole numbers $\longrightarrow$ fractions $\longrightarrow$ algebra

In whole numbers, the possibility of engaging explicitly in generality does not exist except for the laws of operations above. For example, try to express the multiplication algorithm in terms of two arbitrary two-digit numbers $10a+b$ and $10c+d$, (a, b, c, d are single-digit numbers):

$$\begin{array}{rcc} & a & b \\ \times & c & d \\ \hline & ? & ? \end{array}$$

The teaching of fractions therefore interpolates between the imposed concreteness of the teaching of whole numbers and the intrinsic abstraction in the teaching of algebra.

The teaching of fractions can naturally usher in algebra **if it is done properly**.

On one level, one can freely use all four arithmetic operations on numbers only when we come to fractions. To the extent that we must use all four arithmetic operations in algebraic computations, fluency in computations with fractions is a critical foundation for the learning of algebra.

On a deeper level, what we are talking about is the acclimatization of students in grades 4–7 to generality and abstraction. Here is where fractions play a crucial role.

The concept of a **fraction** is itself an abstraction.

Whole numbers are not defined for children in (roughly) grades K–4 and that is not a problem: the canonical mental image of one's fingers can serve to anchor children's thinking about whole numbers regardless of whether there is any definition or not.

Not so for fractions. When fractions are presented as a piece of pizza, severe problems arise. How to think of a fraction such as $\frac{251}{7}$? (This is the origin of the fear of *improper* fractions.) How to multiply two pieces of pizza? What does it mean to divide one piece of pizza by another?

But the tradition of school mathematics is to insist on giving no mathematical definition of a fraction, and rely instead on bringing up different "personalities" of a fraction as the need arises.

For example, one "personality" of a fraction $\frac{a}{b}$ is that it is the totality of a parts when the pizza has been divided into b equal parts, and another is that $\frac{a}{b}$ means "a divided by b", regardless of what that means. No mathematical relationship between the two is mentioned in the standard literature.

It may be conjectured that one reason fractions are difficult to learn is that students have no way of coping with a new "personality" every few days.

It is argued in the education literature that, due to the extraordinarily complex set of understandings surrounding fractions, a precise definition of a fraction would be counterproductive.

However, two facts are worth noting:

(1) If **no definition** has led to non-learning of fractions in elementary school for decades, what is there to lose by giving a definition from which *every single interpretation of a fraction can be logically explained*?

(2) There are many mathematically "rich" concepts with diverse interpretations, but *in mathematics*, each is given a *precise* definition and then all other interpretations are proved as theorems. This is standard practice in mathematics.

Why harp on the need of a definition for fraction?

Because, instead of embracing the opportunity to introduce students to abstraction, gently, by defining a fraction in terms of certain points on the number line, the curricular decision of "NO definition" deprives students this opportunity to confront abstraction for the first time.

Without a definition, no fact about fractions is accessible to mathematical reasoning. Without definitions and without reasoning, fractions is indeed not a learnable subject. Period.

It should not be difficult to carry out education research to verify the causal relationship between bad teaching of fractions and the subsequent non-achievement in algebra.

With a precise definition of a fraction, one **proves** that

$$\frac{a}{b} + \frac{c}{d} \;=\; \frac{ad + bc}{bd}$$

for *all* fractions $\frac{a}{b}$ and $\frac{c}{d}$.

Similarly, if multiplication of fractions is precisely defined, then one **proves** that

$$\frac{a}{b} \times \frac{c}{d} \;=\; \frac{ac}{bd}$$

for *all* fractions $\frac{a}{b}$ and $\frac{c}{d}$.

Observe that with precise definitions and reasoning in place, students get to see, *naturally*, in the course of learning about fractions, how generality is fully expressed by the use of symbols.

Many more examples can be given, but two examples are particularly instructive.

For subtraction, there is the theorem that

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

for *all* fractions $\frac{a}{b}$ and $\frac{c}{d}$ so that $\frac{a}{b} > \frac{c}{d}$.

Here is an example of a general statement which is valid *not* for all $\frac{a}{b}$ and $\frac{c}{d}$, but only for those satisfying the additional condition that $\frac{a}{b} > \frac{c}{d}$. This will teach student that generality does not always means "valid for all things under the sun", and that the *meaning* of each symbol is important.

Another example is the following **Theorem**:
For all fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

This is of course the **cross-multiplication algorithm.** If students are taught this theorem with care, they would know why it is true and why it is important (it is a simple procedure for deciding if two fractions are equal). It is an immensely useful theorem, but it is now equated, in some quarters, with learning by rote.

Without a precise definition of a fraction, this theorem indeed can only be learned by rote. Few students (or teachers) can explain why this theorem is true.

Bad teaching breeds bad learning. Of course.

Without benefit of precise definitions in fractions, students are denied the abstraction and the reasoning that comes with the abstraction. Their level of mathematical sophistication is consequently artificially depressed as they go through the grades instead of being gradually elevated.

Then we throw them to the wolves when they come to algebra, when abstractions can no longer be avoided and a modicum of reasoning is necessary for survival.

Without reasoning, students cannot begin to approach word problems: problem solving is, at the very least, a sequence of logical deductions.

And we ask why students cannot survive algebra?

The transition from arithmetic to algebra
Hung-Hsi Wu, UC Berkeley

This talk will address some of the mathematical aspects of the transition to algebra and will try to respond to some things said in earlier sessions.

Slide Two: Why can't our students achieve algebra? Wu addresses the question at the level of understanding why a typical American student might struggle with learning algebra. The reason given is that they are not being taught the right things and they are not being prepared for algebra.

Slides Three and Four: Take the case of students' inability to work problem in algebra. Wu discusses the problems with using the "key word" approach in instruction ("and, more, etc call for + "of, as much … call for multiplication). The "key word search" habit is antithetical to mathematics as sense-making and detrimental to future learning of algebra. Students are alienated from the words.

[ML: cf connection to Alan Schoenfeld's talk on word problems]

Slide Five: Remainder of the talk will be about teaching of fractions and its impact on the learning of algebra. That being said, Wu frames arithmetic as being about computations with specific numbers and algebra is about generality and abstraction.

Slide Six: Wu compares addition of fractions with addition of rational expressions. In the case of addition of rational expressions, one must have the ability to see each of the terms as a single number for any x.

Slide Seven: Solving algebraic equations requires computing with a number "x" without knowing what it stands for. This is an example of "abstract" thinking. To compute with unknowns requires understanding and using the associative, commutative, and distributive laws. Students have difficulty with working with equations and solving for unknowns because they have not come to understand these laws.

Slide Eight: Curriculum trajectory: Whole numbers, Fractions, Algebra. Wu notes the difficulty of expressing the multiplication algorithm in terms of two arbitrary two digit numbers $10a+b$ and $10c+d$ where a, b, c, d are single digit numbers.

Slight Nine: Wu claims that the teaching of fractions is critical because it is intermediate to the imposed concreteness of the teaching of whole numbers

[ML: cf potential connection to early algebra approach—imposing concreteness on arithmetic is not necessarily helpful in supporting the transition to algebra]

and the intrinsic abstraction in the teaching of algebra.

Wu positions the teaching of fractions as a natural place to usher in algebra. For one thing, one can freely use all four arithmetic operations on numbers once we get to fractions (because the rational numbers are a field). Fluency in computations with fractions is a critical foundation for the learning of algebra. But at a deeper level, we are talking about "acclimatizing" students in grades 4-7 to generality and abstraction.

Slide Ten: "Fraction" is itself an abstract concept and that this is not addressed in textbooks presents a problem. Wu claims that it is not problematic that whole numbers are not given a precise definition because canonical mental images of fingers and counting can anchor children's thinking about whole numbers. But, the situation is different for fractions. He asks what should be the anchor for 7/13 or for 251/7? Wu locates student difficulty fractions with certain representations (circular area model in particular) being presented as insufficient anchors for the concept of fraction.

Slide Eleven: Wu discusses the approach of introducing fractions through the introduction of various personalities or representations of fraction in different contexts. He makes the conjecture that one reason students find it difficult to learn fractions is that learning about so many different personalities of fractions places a large cognitive burden on students.

Slide Twelve: Wu takes issue with the argument from the education literature that it is more productive for students to be introduced to fractions through multiple representations than through a unifying definition for fractions. He notes that (1) Since students already find learning fractions difficult, why not try an approach involving a definition, which can unify the multiple interpretations of fraction that students encounter (a question of "what do we have to lose?") and (2) in mathematical practice, when working with "rich" concepts with multiple interpretations, one gives each a precise definition. Then you can choose one interpretation and deduce all the others as consequences.

Slide Thirteen: Wu feels strongly about the need for the definition of fraction as points on a number line. He argues that this is a crucial opportunity to introduce students to abstraction for the first time.

Slide Fourteen: With a definition, it then becomes possible to prove to students how to add and multiply any two rational numbers in general. This provides a way to see how generality is expressed in symbols in the course of learning about fractions.

[ML: Seems like here there is a connection to one of the early algebra perspective approaches of introducing "algebra as generalized arithmetic" and Saul's emphasis on paying attention to the inductive/ deductive aspects of arithmetic and algebra]

Slide Fifteen: The need in the case of subtraction of rational numbers for the additional constraint that $a/b > c/d$ is an opportunity to teach that the meaning of symbols in general expressions is important. Additional constraints still need to be imposed on statements expressed abstractly.

Slide Sixteen: Wu discusses the cross-multiplication algorithm as a simple, but important procedure for deciding when two fractions are equal. We'd like students to be able to use this procedure with understanding. To do this, a precise definition is required.

Slide Seventeen: The suppression of abstraction in the early grades sets up an artificial and large divide between arithmetic and algebra.

Comments from the conference participants:

Question One: What definition of a fraction do you recommend for a 4th grader?

What is a definition appropriate to a 4th grader. Familiarize children with terminology and basic properties. Collect data and have some hands on experience. The theorizing doesn't come into later. The teachers that Wu works with introduce fraction as point on a number line [to 2nd graders and to 1st grade even] it helps the students to do simple things: adding simple fractions. The serious teaching of fractions begins in grade 5. For example, addition and subtraction of fractions occurs in grade 5. In grade 6, students extend to multiplication and division of fractions. With the number line, the size concept is embodied in the distance from the origin.

Question Two: Is there a chronological age when a normal child is capable of abstract thinking??

Children are capable of much more than people have believed. Piaget's idea of developmental appropriateness not appropriate. Wu remarks that in the Russian system letter notation (x,y,z's) started in 3rd grade. Do not like the pseudo-abstraction with unknowns and blanks—why don't we put in x (like in the Russian system)? What's the difference??

Question Three: Is everyone ready for algebra at grade 8?

Wu: What you decide children can do? International comparisons show us that developmentally children can do algebra at grade 8.
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Civil Rights and The Algebra Project

I was a graduate student in the philosophy department of Harvard in 1957 when the Russians launched Sputnik and our nation responded with what became known as the new math. Then, as now, efforts to change *what* math is taught and how it is delivered are inextricably linked to larger national goals.

Family circumstances pushed me out of graduate school in 1958 and I landed at the Horace Mann School in Riverdale, New York teaching math to 7th and 8th graders for the next three years, trying out the fit between the nation's Sputnik response and my decidedly upper middle class college bound 7th graders, by moonlighting copies of SMSG and Max Beberman materials for the offices of Professor Fehr, head of the math department at Columbia's Teacher's College.

Willard Van Orman Quine, the logician of mathematics, was the star of Harvard's department of mathematics in the 1950's and lo, here was the Quinian construction of integers as equivalence classes of natural numbers deconstructed by Beberman for 9th graders. Looking back, it helps to have a sense of humor about this "stuff" as it is called, and some at least of the habits of mind it encourages or vice-versa, as the case may be.

In the 1950's, C. Vann Woodward rose to become the nation's storyteller of the "stuff" of its history with a readable and mercifully slim book he named "The Strange Career of Jim Crow." The Jim Crow laws, he observed "unlike feudal laws, did not assign the subordinate group a fixed status in society." They were constantly pushing the negro further down. "Its spirit", he argues, "is that of an all-absorbing autocracy of race, an enemy of aggrandizement which makes, in the imagination of the white man, an absolute identification of the stronger race with the very being of the state."

Delivered at first as the James W. Richard lectures at the University of Virginia before unsegregated audiences in the fall of 1954 in response to the Supreme Court's decision on the 17th of May in which Chief Justice Warren concluded that "in the field of public education, the doctrine of 'separate but equal' has no place." The "Strange career" was revisited a decade later after Jim Crow met its "nemesis." Woodward tells it this way:

> On February of that year (1960) four negro college boys, freshmen at the Agricultural and Technical College in Greesnboro, North Carolina, asked politely for coffee at Woolworth's lunch counter and continued to sit in silent protest when refused. The "sit-in" nemesis of Jim Crow was born.

The "sit-in" movement pulled me away from teaching math and from equivalence classes of natural numbers and into living history and the construction of new equivalence classes of constitutional people.

John Doar, Burke Marshall, and Judge John Minor Wisdom were unlikely and unsung heroes participating in the new construction. Doar, in his eighties, is still kicking it and I dropped by his law office in New York a week ago Tuesday to check out its story line, how the sit-in movement, what I have come to think of as an "earned insurgency" against Jim Crow, contributed to a constitutional reach that seeks to make permanent a new equivalence class of voters. I wanted to check the story-line with Doar because I think it germane to the work of the Algebra Project, which works to connect and "earned insurgency" of students in math classrooms to a constitutional reach that seeks to make permanent new equivalence classes of public school students.

Constitutional reach about voting was provided in the written opinion of Judge Wisdom who presided over a three judge panel of the 5th circuit court of appeals sitting in Baton Rouge in the case of United States vs. Louisiana in December 1963 less than a month after President Kennedy was assassinated in Dallas. Judge Wisdom took into account Louisiana's own version of Jim Crow in crafting his finding against the use of literacy tests to qualify voter registration applicants and when Nicholas Katsenbach, the Attorney General, sat in front of the House Judiciary Committee in 1965, to testify for the Voting Rights Act, he was armed with the writings of Judges Wisdom, Tuttle, and Reeves from the 5th circuit, opinions which laid a foundation for the concept that a constitutional person, as a citizen of the nation, may look to the national government for protection of the right to vote--the idea that we participate in this fundamental exercise of democracy, not primarily as citizens of a state, but as citizens of the nation.

These opinions from the 5th circuit were based on a mountain of work done by the Civil Rights Division of the Department of Justice under the Kennedy administration where Bobby Kennedy was Attorney General, Burke Marshall was assistant Attorney General for Civil Rights and John Doar directed the work of the divisions' lawyers in the field.

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The channeling of the "sit-in" movement into an "earned insurgency" worked the demand side of the right to vote. Constitutional reach and changes in National Policy don't occur in a vacuum.

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James Bryant Conant, president of Harvard from 1933 to 1953, he of SAT and ETS renown, put the conundrum this way in his little book, "Slums and Suburbs" published in 1961 as I headed to Mississippi to work on voter registration:

> After the civil war, the people of the United States through their duly elected representatives in congress acquiesced for generations in the establishment of a tight caste system as a substitute for negro slavery. As we now recognize so plainly but so belatedly a caste system finds its clearest manifestation in an educational system.

In 1991, I was back in Mississippi doing the algebra project in middle schools in the

Delta and in Jackson. In 1996, I followed 8th graders from middle school into Lanier, their feeder high school. They had completed an algebra I course and I was looking for a soft landing in geometry. I taught a full load at Lanier for the next 10 years and in 2002 began what has become the concept of forming a cohort, a class of 9th graders from the bottom quartile who commit to taking four years of math every day in block schedule 80 to 90 minutes per day. The idea that these students should not acquiesce in remediation but should work to accelerate their math so as to jump through the nation's three hoops: the algebra hoop, the ACT hoop or the SAT hoop, and the university hoop--that they should have the option to go to college, they should not remediate math in college, and math should not be an obstacle to a career. I think of the process of establishing such cohorts in the bottom quartile of targeted schools across the nation as a way to feed an earned insurgency against our educational caste system with eventual constitutional reach to the concept that a child, a constitutional person, a citizen of the nation, may look to the national government for protection of the right to a quality public school education--the idea that if we participate in this fundamental exercise of democracy, not primarily as citizens of the state, but as citizens of our nation.

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> After the civil war, the people of the United States through their duly elected representatives in congress acquiesced for generations in the establishment of a tight caste system as a substitute for negro slavery. As we now recognize so plainly but so belatedly a caste system finds its clearest manifestation in an educational system.

In 1991, I was back in Mississippi doing the algebra project in middle schools in the

Delta and in Jackson. In 1996, I followed 8th graders from middle school into Lanier, their feeder high school. They had completed an algebra I course and I was looking for a soft landing in geometry. I taught a full load at Lanier for the next 10 years and in 2002 began what has become the concept of forming a cohort, a class of 9th graders from the bottom quartile who commit to taking four years of math every day in block schedule 80 to 90 minutes per day. The idea that these students should not acquiesce in remediation but should work to accelerate their math so as to jump through the nation's three hoops: the algebra hoop, the ACT hoop or the SAT hoop, and the university hoop--that they should have the option to go to college, they should not remediate math in college, and math should not be an obstacle to a career. I think of the process of establishing such cohorts in the bottom quartile of targeted schools across the nation as a way to feed an earned insurgency against our educational caste system with eventual constitutional reach to the concept that a child, a constitutional person, a citizen of the nation, may look to the national government for protection of the right to a quality public school education--the idea that if we participate in this fundamental exercise of democracy, not primarily as citizens of the state, but as citizens of our nation.

The Chicago Mathematics and Science Initiative
Mary Jo Tavormina, Elementary Mathematics Manager

See http://www.cmsi.cps.k12.il.us for more information about the Chicago Mathematics and Science Initiative.

Introduction and Background:
The CMSI (Chicago Mathematics and Science Initiative) focuses on improving teaching and learning of mathematics and science K-8 in Chicago public schools. Chicago has site-based management (schools make their own decisions about curriculum, etc).

Chicago is the 3rd largest district in the country. Chicago Public Schools are engaged with an initiative to improve mathematics and science instruction. The Chicago Public School strategy is to build teacher capacity, use high quality materials, provide in-school supports, and provide timely assessment data that can be used to enhance overall quality.

A few big ideas that frame the CMSI are (1) aligning tools and resources, (2) high quality support, and (3) accountability mechanisms. A major focus is a quest for coherence. CMSI is committed to the idea that developing an understanding of both content and processes of mathematics are essential for students.

One of the goals for Chicago Public Schools is an algebra initiative to provide access to an algebra 1 course at the 8th grade level. The philosophy is that if students are prepared for algebra, they should be granted access to it.

Tavormina talks about the need to ensure that coherent policies are in place. This involves taking apart questions such as: What is algebra? What does it mean to teach algebra? What does it mean to learn and understand algebra? How do we prepare students so that they do not make critical mistakes which present obstacles later on?

One strategy employed by the district is to increase the number of teachers prepared to teach algebra. They are trying to move towards "departmentalization." On this topic, there have been discussions around the necessary qualifications of teachers who teach algebra 1 in the middle grades.

Addressing the issue of "accountability mechanisms", the city of Chicago will have an exit exam in place as of next year to certify what students do in fact know about algebra after 8th grade.

In terms of supporting teaching and learning, some teacher supports that are in place are various forms of PD, coaching, middle grade cohorts. They have found that instructional materials really do bring coherence to what you can do with teachers in PD. There have been conversations about Lesson Study in Japan and contrasting what typically happens in US middle schools with counterparts in Japan.

Some questions that are raised by the work in Chicago:

- How do you know if students are algebra ready?
- What is really working inside classrooms?
- How is algebra initiative connected to student learning?
- How can we support development of effective teaching practices in algebra I?
- What are the other tools that will help teachers to move forward?

Discussion/Questions after the presentation:

With respect to the exit exam, what happens if students don't pass exit exam?

The exam is the decider about whether students move on to the next class after algebra. Students can receive A's and B's and fail the exit exam for algebra. This means they have to re-take the class and/or exit exam. One of the reasons for the standard city-wide exit exam is to make clear what learning expectations are, to reduce confusion from parents and to set up a coherent system.

What percentage pass?

The exam was only piloted last year. In the piloted version, 30% passed. We're trying to have it coherent. Do students "know" algebra? Again, the goal of the exam is to try to make expectations clear and uniform.

Some background from the CMSI Website:

The mission of CMSI (from their website)

The Office of Mathematics and Science supports all mathematics and science instruction in Chicago Public Schools with the vision that high-quality, standards-based mathematics and science experiences, as framed by national, state, and local standards, can be provided to all students. This vision is embodied in the Chicago Math & Science Initiative, the CPS plan to transform mathematics and science instruction by providing coherent programs, more support, and better preparation to enable high quality teaching and thus improved student achievement.

Letter from Chandra James, Director of Mathematics and Science, Chicago Public Schools

Welcome to the Office of Mathematics and Science!

Our expectation is that all students will receive a challenging mathematics and science educational experience so that they are well prepared for future academic and career success.

A challenging mathematics educational experience involves more than mastery of basic arithmetic skills. It includes all mathematical content areas-number sense, measurement, algebra, geometry, data analysis, and probability- and builds upon skills and understandings, from concrete to abstract. Mathematical thinking is developed when we approach mathematics as problem solving, mathematics as communication, mathematics as reasoning, and mathematics as connecting to other content areas and to real-world applications.

All students have a right to quality science instruction. Meaningful experiences in the science classroom involve key science processes: the collection and use of evidence, the thoughtful analysis of data, which is then effectively communicated. Understanding, rather than memorization, is the goal because science is not only content, but also process. Experiencing the processes of investigation, analysis, and documentation is key to really understanding science content, and is enabled through active, inquiry-based teaching.

As important as what students learn is how mathematics and science are taught. The "how" is key to students' understanding. The collaborative nature of scientific inquiry needs to be reinforced with collaborative team work. Successful science experiences use the tools and technology of science and foster curiosity, reward creativity, encourage questioning, and value healthy skepticism - all so that beliefs can be changed as new evidence is presented. A great deal of what is required of good science instruction applies to good mathematics instruction. Effectively teaching both subjects involves hands-on experiences.

We have every confidence that all students can learn challenging curricula. High expectations shared by teachers, family members, the community, and students themselves will help transform student achievement. All students include girls and boys, students of different cultures and ethnicities, linguistic diversities, socio-economic levels, and varied learning styles. All students represent the richness of Chicago's urban environment.

An excellent mathematics and science education will enhance students' lives and our community. Please join us in bringing world-class mathematics and science instruction to our Chicago public schools.

Comments and Reflections on Session 2.3:

Comment One (Susan Jo Russell):
Bob Moses shares that Susan Jo Russell (from TERC) was one of the math coaches in his daughter's first grade classroom in Cambridge, MA. SJR comments that the algebra project has been implemented at elementary, middle, and secondary levels [though it has focused its efforts primarily at the secondary level]. She asks whether RM would say something about what he is seeing at the secondary level.

RM reflects on SJR's session and three things he noticed. He says that there seems to be a convergence that is growing and that we should nurture about how to teach the mathematics. There is a convergence around pedagogical approach. For example, the frame of the algebra project is experiential learning. There is some story, some "word problem" and some reflection. Then, there is some conceptualization of those reflections and applications of them. This activity could be entered into at any stage [elementary, middle, high]

RM explains that one thing that the AP has been especially attuned to lately has been issues of mathematical discourse. There is a divide between the ordinary language that people speak and mathematical conceptual language, which is not natural spoken language. For an easy example, RM gives the comparison "Herb is taller than Bob." He explains that in this sentence the feature you're after is height. So, we need to reconstruct this sentence starting off with the height of Herb. We can rephrase this in a way that is more transparently mathematical as "The height of Herb is greater than the height of Bob." In the original statement, where is the information about height? It is encoded in the syntax "is taller than." In the reformulation, it is encoded in a different way.

[This point seems connected to Guershon Harel's discussion of the difficulties students have with representing given information in mathematically useful ways. It also seems connected to Alan Schoenfeld's talk on student difficulties with word problems].

So, being able to participate in mathematics involves making a grammatical shift and talking about language in relation to math. Moses claims that the hard work is not actually in the symbols, but in actually getting the recoding of what we say in ordinary language into conceptual language—which *then* gets symbolized.

RM talks about how introducing a level of notation that is completely owned by the students (i.e. invented by the students for particular purposes) has been very useful. RM notes that student difficulties with understanding and encoding quantified statements may also be rooted in linguistic difficulties. He suggests that it may be helpful to think about variables as serving the same kind of function as pronouns serve in relation to names in spoken language.

This disconnect between natural language and conceptual language is what RM refers to as the different between "feature talk" and "people talk."
For example, "feature talk" [the height of Herb is taller than the height of Hob] as

opposed to “people talk” [Herb is taller than Bob]

Two other examples:

- RM had an example where students were looking at subtraction families: 50-0, …. 50-(-10). In the discussion, saw a shift in students' metaphor for subtraction to the “removal metaphor”. Didn't hear any of that any place else. RM identifies a need for the students to recalibrate their metaphors for add and subtraction as they move from arithmetic to algebra.
- How does direction get into number? The notion of direction resides in addition and subtraction in arithmetic. As soon as we attach direction to number, it means that students have to recalibrate what they're thinking about with respect to addition and subtraction.

Participant comment for Wu: Could you tell us about the use of the number line in international and US work?

Wu answers that the Singapore curriculum does use the number line. He says it is useful because it reifies the abstract notion of a fraction. Marsha L. from the Lawrence Hall of Science suggests that Wu show a number line with fractions on it and share in concrete terms how the number line could be useful for supporting student understanding of rational number. Wu takes the opportunity to clarify some of his earlier statements about the number line.

Wu begins an explanation: What is m/n? Let's ask for 1/5, 1/7…1/n? Everything comes down to knowing what you're unit is. He explains that he finds the “pie approach” (circular area model approach) is completely no good because fractions belong to numbers—the pieces in each piece of pie mean the AREA of the piece of pie.

With the number line approach: 1/5 is located at the first tick-mark of a line segment of unit length divided into five equal parts. Now, if you want to find 2/5, 3/5, … you just march 2, 3, … times. The advantage to this approach is that there is no distinction between proper and improper fractions. Wu argues that making something like 25/7 meaningful is more difficult with a pie model.

The “marching along the number line” to find m/n also gives a clear way to compare fractions. Larger fractions are to the right of smaller fractions on the number line. If you want to convince a child that 4/5 is bigger than 2/3, you just need to do the experiment [march along the number line two distances of length 1/3 or four distances of length 1/5].

David Carraher comments that he understood the logic and that this was a definition, but that wasn't it true that this is a way to *model* what rational numbers are. That it is not literally true that a point on a line is a number. “A number is a point on a line” is a metaphor for number, not literally a statement about what numbers actually are. It is a question about the distinction between the representation and the idea, which is not observable.

Wu responds with a remark about the practice of choosing the canonical representative for a particular fraction (e.g. 1/5, 2/10, 4/20, … all represent the same idea but we'd typically choose to express this as 1/5 by default unless there were some other compelling reason to choose a different representative). Of course all of these "points" are located at the same place on the number line (1/5, 2/10, 4/20, …) and this is something we'd want students to understand about what fractions are and how we work with them.