Preparing teachers to teach algebra

Potential remedies

• Teacher retention.
• Bringing teachers back.
• More careful attention to assignment to subject areas.
• Incentives
Preparing teachers to teach algebra
Three part problem

• Preparing high school teachers who will teach algebra.

• Preparing middle school teachers who will teach algebra.

• Preparing elementary school teachers to teach algebra (Early algebra)
Preparing High School Teachers

• Preparation, typically, requires math major.
• Good news: More years of more students.
• Problem to create large enough numbers:
  – Some states are exporters, some importers, some from other countries.
  – Uteach model, …
• Prepare to teach students required to take courses.
• Concerns about their algebraic skills.
Preparing Middle School Teachers

• Depends on state certification, typically, no special middle school certification.
• Either secondary or elementary certified.
• Concerns about the mathematics background of the elementary certified.
Initial impetus

- Montgomery County approaches Mathematics Department.
- Mathematics Department involves EDCI.
- Desire to have 80% of all students take and pass Algebra 1 by end of 8th grade by 2010.
- Need highly qualified teachers ready to teach Algebra and Geometry in Middle Schools.
- (Algebra I in MD includes both Algebra and Data Analysis content)
Who the teachers are, why they enrolled

- I think it was the kids, cause the kids were so...like, they’d ask me questions, and I didn’t know, but I’d want to find out for them. And they were willing to work and find out and come tell me. They’d come and say, “Look at what I’ve found Ms. ___” I would look at it, and it was a little 9 year old explaining stuff to me...
Who the teachers are, why they enrolled

• I remember thinking, “you know what? I always wanted to go back and learn more math.” I wanted to take calculus, I really want to know how all that stuff works. And I thought, well, this might be a good place to start. And then I happened to interview for this other job, and now I’m a middle school math teacher
Who the teachers are, why they enrolled

• I’m really convinced at this point that if you don’t have a deep understanding of the mathematics yourself, you’re not gonna teach it very well. I thought before, “Well, I have the textbook. I can just follow that and the kids ‘ll be fine,” but it’s really way more complicated than that. And that’s why I joined the Masters program in the first part ‘cause I wanted to learn more Math – just content, but now I can see where if you don’t understand it all you can do is stand and deliver.
Milestones

Fall 03  Initial contacts from MCPS to university
04       Design process
April 05 University approval of degree program
June 05 First cohort enters program (17)
March 06 Support for two cohorts from Maryland
           Higher Education Commission
June 06 Second cohort enters program (23)
Feb 08  Support for first Prince Georges County
        Cohort from Maryland Higher Education
        Commission
April 08 First PGCPS cohort begins (22)
May 08  First cohort of fourteen graduates
Overview of initial program design

• 30-credit, 3-year M. Ed. Program.
• 3 mathematics education courses;
• 3 mathematics courses;
• 3 integrated mathematics and mathematics education course;
• 1 educational inquiry course.
First Calendar year: The algebra cycle

- EDCI 650 Trends in Mathematics Education
- EDCI 655 Algebra in the Middle School Curriculum
  
  (1st Integrated Course)
- Math 480: Algebra for Middle School Teachers
Second calendar year: Data Analysis Cycle

• EDCI 657 Understanding and Engaging Students’ Conceptions of Mathematics
• EDCI 656 Teaching and Learning Statistics and Data Analysis in the Middle School
• Math 481 Statistics and Data Analysis for Middle School Teachers
Third calendar year: Geometry cycle

- EDCI 654 Assessing Mathematical Understanding
- EDCI 688c Teaching and Learning Geometry in the Middle School
- Math 482 Geometry for Middle School Teachers
EDCI 696, Conducting Research on Teaching

• Action research project
• Classroom based questions
• Infused throughout the final calendar year
Additions as part of first MHEC proposal: Connections to the Field

• Visiting other cohort participants classrooms
• Being visited by another
• Videotaping your classroom
• Observations from UMCP and/or county personnel
Addition by Eden Badertscher: Mathematical Inquiry Strand

• Influence by a philosophy of mathematics course and a doctoral mathematics course.
• Inquiry in mathematics; inquiry about mathematics.
• Short experiences over five semesters.
Additions for PGCPS

• In addition to inquiry strand, strands on Culturally Relevant Pedagogy, English Language Learners, and Special Education.

• Focus on schools “In improvement”

• School level commitment: groups of teachers, administrators attend *Lenses on Learning* program, Math Department support for after school activities.

• (Challenge: Some teachers from Philippines, West African, and India.)
EDCI 655: Teaching and Learning Algebra in the Middle School
Course objectives

• To gain enhanced understanding of the mathematics of school algebra
• To gain insight into the critical learning challenges that algebra students face
• To gain understanding of various pedagogical models for teaching school algebra
• To develop skill in applying knowledge about mathematics teaching and learning to lesson planning and classroom practice
Course projects/requirements

• Case analysis of student learning
• Lesson study
• Curriculum analysis
• Active class participation
Sample mathematical questions addressed

• What is school algebra?
• What are some different approaches to algebra instruction?
• Why is \((-1)(-1) = +1\)?
• Why do we switch the direction of the inequality when multiplying or dividing by a negative number?
• Why do we distribute when multiplying polynomials?
• How/why do the exponent rules work?
Math 480: Algebra for Middle School Teachers
Focus

Topics covered in a college algebra course:

• Number systems
• Functions
• Equations and inequalities
• Divisibility of numbers and polynomials
Different vantage points and encouraged student connections.

• A broader perspective

  Example: Lines are sometimes represented as \( y = mx + b \), and sometimes as \( ax + by = c \). Which types of situations are most naturally represented by which type of equation? What is the impact of changing one of the parameters? What information is most easily visible in each form?

• A historical perspective

  Example: Al-Khwarizmi's approach to solving "A square and ten roots make 39" involved actually "completing the square" geometrically. How does this approach correspond to the algebraic procedure taught now? What are the advantages and disadvantages of using a concrete model?
Algebra Inquiry Strand Task

• Investigate the set of points that are equidistant from a line and a point not on that line
  - What does this definition mean?
  - Find the set of points?
  - Raise questions or conjectures about points

• Investigate questions and conjectures and raise new ones based on work
Discussion: “What is a proof?”

Many students wanted to prove that the set of points was a parabola; some tried symbolic manipulations, but many tried using specific points. We discussed what constituted a proof, and what different approaches could and could not offer, and whether proof established Truth.
Some observations about design considerations

• Having the curriculum related courses keeps the program job-related for the teachers.
• The mathematics is seen as challenging, but teachers feel a sense of accomplishment on completion.
• The inquiry experience adds an important dimension to how they think about themselves with respect to mathematics.
• There is more mathematics instruction than in initial elementary certification and significantly more work on instruction, with people who are experienced teachers.
Final thought

• What about other sorts of teacher knowledge?
  – Like capacity to communicate across difference to help students become intentional learners (Lampert, 2001) of school mathematics.

• Is a teacher equally effective in different school contexts?
Preparing teachers to teach algebra
Dan Chazan, University of Maryland

Introduction:
DC starts with the statement that we need to think programmatically about preparing teachers to teach algebra. A large part of the issue doesn’t concern content details, but more on the context of teaching in the US.

He notes that for some time, more people are leaving teaching than entering teaching. [See slide 2]. The NCTAF projects that the situation will become even more dire in the next 5-10 years. When one thinks of this issue of preparing teachers to teach algebra, one needs to think about general problem of finding teachers to teach in schools. This is especially true in high need areas. One potential strategy is to focus on teacher retention (not just recruitment) and also thinking about mechanisms to bring back to the teaching force people who have left teaching.

In thinking about the problem of finding teachers to teach in schools, there are three main fronts [see slide 5]. We need to be thinking about preparing elementary teachers to teach early algebra. How do we make sure teachers are prepared to do that work? We also need to be thinking about this issue at the middle school and high school levels. In this talk, DC will focus at the middle school level but will first make a few remarks about the preparation of high school math teachers.

A high school teaching credential usually requires a math major. It is good news that more students are taking mathematics now, because this means that there will be more jobs for math teachers. The flip side of this is that the need is also increased. The situation is also quite complicated at the state-by-state level. States are highly variable in whether they import or export potential mathematics teachers. DC notes that Maryland is an importer of teachers from Penn and also from all around the world. (Slide 6)

We need to keep in mind issues of numbers and how to prepare. Now, another thing to note about the situation with algebra in many places at this time is that now teachers need to teach kids who are required to be in their courses. This changes their work.

There are concerns about the algebraic skills of our graduating math majors—concern that students’ algebraic skills aren’t stronger when they graduate than when they come to college [Was this at UMD?] This is a genuine concern among the faculty in the math department and the implications need to be considered.

In thinking about middle school math teachers, their certification is usually either elementary or secondary since there is typically no middle school teaching credential. Clearly, with increased mathematical demands at the middle school level, the preparedness of those with elementary credentials to meet these demands is a concern. (Slide 7)

To address this concern, DC turns next to discussing a particular program at University of
Maryland, which is designed to work specifically with elementary certified math teachers and preparing them to teach algebra (and other topics). The initial impetus for the project was Montgomery County School district approaching the mathematics department at UMD. Their goal is to have 80% students taking and passing algebra by end of 8th grade by 2010. Obviously in order to achieve this, they will need highly qualified teachers ready to teach algebra and geometry in middle grades.

The participants in this program had concerns about their own mathematical knowledge. DC notes that they are people who are already teaching and quite committed. By investing in them, we are keeping them in teaching. [Slides 11 and 12 have quotes from some of the participants in the course about what motivated them to take the course].

The program is a 10 course masters degree program running in three cycles (algebra cycle, data analysis cycle, and geometry cycle). The program is a 30 credit 3 year M. Ed. Program which includes 3 math education courses, 3 mathematics courses, 3 integrated mathematics and math ed courses, and 1 educational inquiry course. The courses in the mathematics department follow the courses math ed. They hope that this way the course in the math department has a better chance of being successful. The integrated content and pedagogy course is focused around the district’s curriculum.

In addition to the math and math ed courses there is a course on “conducting research on teaching”. It involves an action research project, and engagement in classroom based questions. These activities are infused throughout the year. Part of the design of the program was to have more connections to the field. Participants would visit the classrooms of other cohort participants and be visited by others. They would videotape their own classrooms. This would be in addition to observations from UMCP and the county.

DC talks about the contribution to the design of the program made by Eden Bedertscher who was writing her doctoral dissertation in math ed at UMD. She was influenced by a philosophy of mathematics course and a doctoral mathematics course she had taken. She wanted the participants in the program to have some experience engaging in “mathematical inquiry.” So, the idea was to adapt the program and have participants engage in short inquiry experiences over five semesters.

In addition, the original design of the program has been adapted to include strands on culturally relevant pedagogy, ELL, special education, and also a focus on schools who are “in improvement.” So, the program particularly targets to work with particular schools and have school level work (e.g. after school activity).

DC turns to discussing a bit about the course “Teaching and Learning Algebra in the Middle School.” The course activities include generating lists of perplexing questions in algebra they’d been asked as teachers. They wanted teachers to explicitly talk about student conceptions and misconceptions (e.g. around variable, equality, graphs, etc). Course projects included cases analyses of student learning, a lesson study experience (designed and carried out in a local school), and analyses of algebra curriculum. After
this course, the student would then take an algebra course in the mathematics department.

In the teaching and learning of algebra course, some sample questions addressed might be: (1) What is school algebra? (2) What are some different approaches to algebra instruction? (3) Why is (-1)(-1)=+1? (4) Why do we switch the direction of the inequality when multiplying or dividing by a negative number? (5) Why do we distribute when multiplying polynomials?

In the algebra course in the math department, students discuss (1) number systems (2) functions (3) equations and inequalities, (4) divisibility of numbers and polynomials. Different vantage points are taken and it is important to encourage students to make connections between the different vantage points.

In the algebra inquiry strand, an example investigation might be: Investigate the set of points that are equidistant from a line and a point not on that line.

- What does this definition mean?
- Find the set of points
- Raise questions

[See the slides for some other examples]

A discussion that comes up naturally in this strand of the program is “What is a proof? They have found that the algebra strand was a good entry-point for bringing up these issues.

DC makes some observations about design considerations:
- Having curriculum related courses keeps the program job-related for the teachers
- The math is seen as challenging
- Inquiry experience added important dimension around teacher’s identity development as they came to see themselves as active doers of mathematics
- The opportunity to work with teachers in far greater depth than in pre-service is seems like a promising strategy for retention and professional development.
- This program may be offered as an exemplar of how education and mathematics departments could work together.

Final thoughts:
What other sorts of teacher knowledge need to be developed. There are all sorts of other teacher knowledge far less clear to provide. See Lampert (2001). How do we help the participants help students become intentional learners of mathematics? It is not necessarily clear that the preparation in the program necessarily addresses this charge.

**Conference Participants’ Comments and Questions:**
1st audience Question: Will a teacher be effective across contexts? Can we just move effective teachers around?

Niral Shah, UC Berkeley:
Has the program design team thought about having teachers think about the math beyond middle school?

DC response:
This is a part of the concern of the course. Provide people with more than what they need in their teaching. There was certainly a consciousness of going beyond—but not too far.

Amy Cohen, Rutgers:
There are clearly some benefits of working with mature, motivated teachers. Can you suggest any lessons that might be relevant to pre-service teachers?

DC response:
Do we want to create a concentration for elementary pre-service teachers who will be at middle school in order to bolster their mathematics skills?

Steffen, University of Wisconsin:
Do you have any data wrt when you require middle school math teachers to know more math from the start?

DC response:
One of the ramifications of NCLB—have to articulate if teachers are highly qualified. This is part of what pushed the county to come to the mathematics department first.

Chicago (Mary Jo Tavormina?):
In Chicago, one does need a middle grade endorsement to teach. They are also aiming to provide more experiences with middle school math beyond what you’re teaching. Investigate the highly qualified requirements—pretty stringent. This is always pushing the states forward. What do teachers need before they are coming in? How are we continuing to develop teachers’ knowledge as you’re in schools?
ALGEBRA FOR TEACHING
SOME RECOMMENDATIONS FOR TEACHER PREPARATION

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MSRI Critical Issues, 2008
Outline

1. In a Typical Day...
   - Algebra in algebra class
   - Algebra in non-algebra classes
   - Algebra outside of class

2. Some Recommendations
   - For Abstract Algebra
   - For Linear Algebra
   - For Number Theory
   - For Other Things

3. Conclusion
In algebra class

- Is \( \frac{x^2 - 3x}{x^2 - 9} \) the same as \( \frac{x}{x+3} \)?
IN ALGEBRA CLASS

- Is \( \frac{x^2-3x}{x^2-9} \) the same as \( \frac{x}{x+3} \)?
- When is \( x^n - 1 \) a factor of \( x^m - 1 \)?
IN ALGEBRA CLASS

- Is $\frac{x^2-3x}{x^2-9}$ the same as $\frac{x}{x+3}$?
- When is $x^n - 1$ a factor of $x^m - 1$?
- Can a quadratic equation have more than 2 roots?
In a Typical Day...

Some Recommendations

Conclusion

**IN ALGEBRA CLASS**

- Is $\frac{x^2-3x}{x^2-9}$ the same as $\frac{x}{x+3}$?
- When is $x^n - 1$ a factor of $x^m - 1$?
- Can a quadratic equation have more than 2 roots?
- At how many values do two polynomials have to agree before we can say they are equal?
In a Typical Day...

In algebra class

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- When is \( x^n - 1 \) a factor of \( x^m - 1 \)?
- Can a quadratic equation have more than 2 roots?
- At how many values do two polynomials have to agree before we can say they are equal?
- And what does it mean for two polynomials to be equal?
IN ALGEBRA CLASS

- Is \( \frac{x^2-3x}{x^2-9} \) the same as \( \frac{x}{x+3} \)?
- When is \( x^n - 1 \) a factor of \( x^m - 1 \)?
- Can a quadratic equation have more than 2 roots?
- At how many values do two polynomials have to agree before we can say they are equal?
- And what does it mean for two polynomials to be equal?
- Can a system of linear equations with integer coefficients have an irrational solution?
In a Typical Day…

Some Recommendations

Conclusion

**In algebra class**

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- At how many values do two polynomials have to agree before we can say they are equal?
- And what does it *mean* for two polynomials to be equal?
- Can a system of linear equations with integer coefficients have an irrational solution?
- Can a system of 3 linear equations in four unknowns have exactly 3 solutions?
In algebra class

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- At how many values do two polynomials have to agree before we can say they are equal?
- And what does it mean for two polynomials to be equal?
- Can a system of linear equations with integer coefficients have an irrational solution?
- Can a system of 3 linear equations in four unknowns have exactly 3 solutions?
- If a polynomial doesn’t factor over \( \mathbb{Z} \), can it factor over \( \mathbb{Q} \)?

(McCallum)
**IN OTHER COURSES**

- Why is arithmetic with complex numbers like arithmetic with polynomials?
In OTHER COURSES

- Why is arithmetic with complex numbers like arithmetic with polynomials?
- What does it mean for two functions to be equal?
In other courses

- Why is arithmetic with complex numbers like arithmetic with polynomials?
- What does it mean for two functions to be equal?
- How can I find a polynomial that agrees with a table?
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- Why do they use $f^{-1}$ for inverse function?
- For polynomials, why does the $h$ in the denominator of

$$
\frac{f(x + h) - f(x)}{h}
$$

always cancel out?
In a Typical Day...  

Some Recommendations

Conclusion

IN OTHER COURSES

- Why is arithmetic with complex numbers like arithmetic with polynomials?
- What does it mean for two functions to be equal?
- How can I find a polynomial that agrees with a table?
- Why do they use $f^{-1}$ for inverse function?
- For polynomials, why does the $h$ in the denominator of $\frac{f(x + h) - f(x)}{h}$ always cancel out?
- **Why** can’t you trisect a 60° angle with a straightedge and compass?
Professional uses of Algebra

The class is using calculators and estimation to get decimal approximations to $\sqrt{5}$. One student looks at how you do out long multiplication and realizes that none of these decimals would ever work, because if you square a finite (non-integer) decimal, there’ll be a digit to the right of the decimal point. So you can’t ever get an integer. She deduces that that $\sqrt{5}$ can’t be rational.

— adapted from “A Dialogue About Teaching” in *What’s Happening in Math Class?* Teacher’s College Press.
PROFESSIONAL USES OF ALGEBRA

Nine year old David, experimenting with numbers, conjectures that, if the period for the decimal expansion of $\frac{1}{n}$ is $n - 1$, then $n$ is prime.

— Adapted from a Reader Reflection by Walt Levisée in the *Mathematics Teacher* (March, 1997).

Speaking of decimals, how would you characterize the “unit fractions” $\frac{1}{n}$ that have terminating decimal expansions? What can you say about the periods of the repeating ones?
How can you help your students understand the “multiplication rule” for complex numbers?

\[ |zw| = |z| |w| \text{ and } \text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w) \]
PROFESSIONAL USES OF ALGEBRA

How can you help your students understand the “multiplication rule” for complex numbers?

\[ |zw| = |z| |w| \text{ and } \text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w) \]

What if they don’t know any trig?
PROFESSIONAL USES OF ALGEBRA

How can you generate “nice” problems, for example:

How big is $\angle Q$?
PROFESSIONAL USES OF ALGEBRA

Or...

\[ f(x) = 140 - 144x + 3x^2 + x^3 \]

Find the zeros, extrema, and inflection points
Suggestions for Abstract Algebra

Instill a sense that algebraic objects are open to experiment
SUGGESTIONS FOR ABSTRACT ALGEBRA

Instill a sense that algebraic objects are open to experiment

- Start with rings, fields, and polynomials—not groups
SUGGESTIONS FOR ABSTRACT ALGEBRA

Instill a sense that algebraic objects are open to experiment

- Start with rings, fields, and polynomials—not groups
- Solve equations in these systems
**Suggestions for Abstract Algebra**

Instill a sense that algebraic objects are open to experiment

- Start with rings, fields, and polynomials—not groups
- Solve equations in these systems
- Develop in detail the structural similarities and differences between \( \mathbb{Z} \) and \( \mathbb{Q}[x] \)
SUGGESTIONS FOR ABSTRACT ALGEBRA

Instill a sense that algebraic objects are open to experiment

- Start with rings, fields, and polynomials—not groups
- Solve equations in these systems
- Develop in detail the structural similarities and differences between $\mathbb{Z}$ and $\mathbb{Q}[x]$
- Construct $\mathbb{C}$ as $\mathbb{R}[x]/(x^2 + 1)\mathbb{R}[x]$
**SUGGESTIONS FOR ABSTRACT ALGEBRA**

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- Construct more general splitting fields in the same way
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  - the theory of equations
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  - the geometry of the regular $n$-gon
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  - roots of unity
  - the geometry of the regular $n$-gon
  - Cardano
  - the theory of SE and compass constructibility
SUGGESTIONS FOR LINEAR ALGEBRA

Help students develop a knack for reasoning by linearity
SUGGESTIONS FOR LINEAR ALGEBRA

Help students develop a knack for reasoning by linearity

\[
\begin{align*}
3x + 5y - 7z + w &= 2 \\
4x - 5y + 10z - w &= 8 \\
6x - 5y + 3z + 2w &= 6
\end{align*}
\]
SUGGESTIONS FOR LINEAR ALGEBRA

• Stress some basics...
SUGGESTIONS FOR LINEAR ALGEBRA

- Stress some basics...
  - extension by linearity
SUGGESTIONS FOR LINEAR ALGEBRA

- Stress some basics…
  - extension by linearity
  - change of basis
Suggestions for Linear Algebra

- Stress some basics…
  - extension by linearity
  - change of basis
- Carefully develop vectorial methods in geometry…
SUGGESTIONS FOR LINEAR ALGEBRA

- Stress some basics...
  - extension by linearity
  - change of basis
- Carefully develop vectorial methods in geometry...
  - including the “extension program” from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$...
Suggestions for Linear Algebra

- Stress some basics...
  - extension by linearity
  - change of basis
- Carefully develop vectorial methods in geometry...
  - including the “extension program” from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$...
  - and the use of matrices to represent linear transformations
SUGGESTIONS FOR LINEAR ALGEBRA

- Stress some basics...
  - extension by linearity
  - change of basis
- Carefully develop vectorial methods in geometry...
  - including the “extension program” from $\mathbb{R}^2$ or $3$ to $\mathbb{R}^n$...
  - and the use of matrices to represent linear transformations
- Connect various uses of determinants...
SUGGESTIONS FOR LINEAR ALGEBRA

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- Carefully develop vectorial methods in geometry...
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- Connect various uses of determinants...
  - as area and volume
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- Connect various uses of determinants...
  - as area and volume
  - as a test for linear dependence
Suggestions for Linear Algebra

- Stress some basics...
  - extension by linearity
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- Carefully develop vectorial methods in geometry...
  - including the “extension program” from \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) to \( \mathbb{R}^n \)...
  - and the use of matrices to represent linear transformations

- Connect various uses of determinants...
  - as area and volume
  - as a test for linear dependence
  - as an algebraic tool (resultants, Cramer, matrix inverses...)
Suggestions for linear algebra

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- Exploit matrix algebra as a formal “bookkeeping” tool...
Suggestions for Linear Algebra

- Stress some basics...
  - extension by linearity
  - change of basis
- Carefully develop vectorial methods in geometry...
  - including the “extension program” from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$...
  - and the use of matrices to represent linear transformations
- Connect various uses of determinants...
  - as area and volume
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- Make connections among eigenvalues, geometry, and algebra
Let general results evolve from numerical experiments
Suggestions for Number Theory

Let general results evolve from numerical experiments

- Compare arithmetic in $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Q}[x]$
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- Talk to Glenn Stevens
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Algebra and algebraic reasoning around topics in the undergraduate mathematics curriculum can help prospective teachers enter the profession with a coherent view of secondary mathematics.

But this doesn’t come for free. Explicit connections to the daily work of high school teaching should be a part of every undergraduate course.
IN CONCLUSION...

This doesn’t mean developing courses in high school mathematics from an “advanced” perspective. It means developing courses that develop the content and methods of undergraduate mathematics while taking seriously the profession-specific needs of high school teachers.
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This doesn’t mean developing courses in high school mathematics from an “advanced” perspective. It means developing courses that develop the content and methods of undergraduate mathematics while taking seriously the profession-specific needs of high school teachers.

**Conjecture:** Such courses would benefit *all* mathematics majors, not just prospective teachers.
Thanks

Al Cuoco (acuoco@edc.org)

www.edc.org/cme
5.01388 cm²
50.1387 cm²
Reflections on Question 3

Al Cuodo, EDC (acuoco@edc.org) and www.edc.org/cme

The theme of the lecture is to highlight some key ideas and areas in the undergraduate mathematics curriculum, which could greatly strengthen the mathematical understanding of prospective teachers and their ability to teach algebra at the middle and high school level.

[ML: Connection to the approach in DC’s course where teachers generate lists of the perplexing questions that students ask them. Of course here, the emphasis is on building teachers’ backgrounds before they become teachers and on making sure they engage at an undergraduate level with the mathematics that gives them perspective on the mathematics they teach at the high school level. So there is a pre-service/in-service distinction and a high school/middle school difference between DC’s approach and what AC is proposing].

Slide One: In algebra
- Is \( x^2-3x/(x^2-9) \) the same as \( x/(x+3) \)?
- When is \( x^n - 1 \) a factor of \( x^m - 1 \)?
- Can a quadratic equation have more than 2 roots?
- At how many values do two polynomials have to agree before we can say they are equal?
- What does it mean for two polynomials to be equal?
- Can a system of linear equations with integer coefficients have an irrational solution?
- Can a system of 3 linear equations in four unknowns have exactly 3 solutions?
- If a polynomial doesn’t factor over \( \mathbb{Z} \), can it factor over \( \mathbb{Q} \)?

Slide Two: In other courses
- Why is arithmetic with complex numbers like arithmetic with polynomials?
- What does it mean for two functions to be equal?
- How can I find a polynomial that agrees with a table?
- Why is \( f^{-1} \) used for inverse function?
- For polynomials, why does the \( h \) in the denominator of the difference quotient always cancel out?
- Why can’t you trisect a 60 degree angle with a straightedge and compass?

Slide Three: passage from “A dialogue about teaching” about a student’s argument for why \( \sqrt{5} \) cannot be rational.

Slide Four: Example of nine year old student experimenting with numbers and conjecturing that if the period for the decimal expansion for \( 1/n \) is \( n-1 \), then \( n \) is prime.

Questions about decimals: How would one characterize the “unit fractions” \( 1/n \) that have terminating decimal expansions? What can you say about the periods of the repeating ones?
Slide Five: How can you help your students understand the multiplication rule for complex numbers (visualize as 2-d vectors). How would you help them if they didn’t know any trig?

Slide Six/Seven: discusses the importance of developing the knowledge necessary to generate “nice” problems. [examples including extrema and inflection points of particular “nice” polynomials]

Slide Eight: Some recommendations for the study of abstract algebra
- Instill a sense that algebraic objects are open to experiment
  - Start the course investigating rings, fields, and polynomials, not groups
  - Solve equations in these systems
  - Develop in detail the structural similarities and differences between Z and Q[x]
  - Construct C as an extension field of R (C as R[x]/(x^2+1))
  - Construct more general splitting fields in the same way
  - Tie Galois theory to
    - The theory of equations
    - Roots of unity
    - Geometry of the regular n-gon
    - Cardano’s formula
    - Theory of SE and compass constructability

Slide Nine/Ten: Recommendations for study of linear algebra
- Help students develop a knack for reasoning by linearity
- Stress some basics
  - Extension by linearity
  - Change of basis
- Carefully develop vectorial methods in geometry
  - Extension program from R^2 and R^3 to R^n.
  - Use of matrices to represent linear transformations
- Connect various uses of determinants
  - As area and volume
  - As a test for linear dependence
  - As an algebraic tool (resultants, Cramer’s rule, inverting matrices)
- Exploit matrix algebra as a formal “bookkeeping tool”
  - In adjacency and scheduling problems
  - As tools for solving recurrence equations
- Make connections among eigenvalues, geometry, and algebra

Slide Eleven: Suggestions for Number theory
- Let general results evolve from numerical experiments
- Compare arithmetic in Z, Z/nZ, Z[i], and Q[x]
- Examine Euclid’s algorithm in Z, Z[i], and Q[x]
Al Cuoco, Reflections on preparing teachers to teach algebra

Session 3.3 Discussion around Question 3

- Make localization and reduction general purpose tools
- Connect Pythagorean and Eisenstein triples to
  - Norms of quadratic fields
  - Rational points on conics
- Develop the theory of repeating decimals
- Connect the Chinese remainder theorem to Lagrange interpolation
- Talk to Glenn Stevens…

Slide Twelve: Suggestions for Other things

- Encourage reasoning about calculations and operations
- The theory of finite differences is useful in teaching
- Polynomial calculus is all algebra
  - The remainder when \( f(x) \) is divided by \( (x - a)^2 \)
  - The Taylor expansion about \( x = a \) is a polynomial
  - \( C \) can be used to derive the addition formulas for sine and cosine (rather than the other way around)
- Summatory polynomials are useful in both calculus and polynomial interpolation
- Function algebra gives an example of algebraic structure
- Statistics has deep connections to linear algebra
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Slide Thirteen: Conclusion

Algebra and algebraic reasoning around topics in the undergraduate mathematics curriculum can help prospective teachers enter the profession with a coherent view of secondary mathematics.

But this doesn’t come for free. Explicit connections to the work of high school teaching should be part of every undergraduate course.

Slide Fourteen

This doesn’t mean developing courses in high school mathematics from an “advanced” perspective. It means developing courses that develop the content and methods of undergraduate mathematics while taking seriously the profession-specific needs of high school teachers.

Conjecture: Such courses would benefit all mathematics majors, not just prospective teachers.

Comments and questions from conference attendees:

Sybilla Beckmann commented on AC’s recommendation for the algebra course. Starting with rings, fields, and polynomials is what they do at Georgia. She also mentions a text by Stillwell that teaches abstract algebra from a geometric approach.

Notes by Mariana Levin
Jean Mitchell noted that in preparing elementary and secondary teachers to teach math, one of the recurring puzzlements is at what point and how during college mathematics are they going to get a “meta-sense” of the discipline. For example, developing some measure of importance of certain things they’re learning for their future work in high school. How can we build this programmatically—the ability to think about “What is mathematics?” as a body of knowledge?

Wu remarks that in the mathematics department, they have a three-semester sequence devoted to the math of the secondary school. The course is designed for future high school teachers. He says that the course covers the kinds of things discussed in the talk and provides the foundations from which they can hope to be good teachers.

Someone comments that the history of mathematics might be a useful requirement for secondary certification. They mentioned that they thought it was useful for adding the historical perspective that Roger Howe had mentioned in his presentation.

Response: History is important for getting the lay of the land and understanding how ideas developed. One of the values of being familiar with a historical approach is that you can develop the material in a way that is faithful to the way it was developed historically. One example is the need to invent or at least work with a new kind of number (complex numbers) to deal with solving third and fourth degree polynomials (cf. Cardano and Tartaglia). The act of making believe that such “complex numbers” existed and behaved in certain ways was revolutionary.

Amy Cohen comments that in the history of math course at Rutgers, students reported coming to understand now why people did proofs. Many students were quite amazed that there were times when some particular piece of knowledge were not known.

Dan Chazan comments on particularly the utility of using primary documents in the teaching of the history of mathematics in the case of pre-service teachers. Teachers should understand mathematical thinking (historically) in a way that has some parallels with how they should understand student thinking.

H. Edwards talks about historical examples of “false proofs” and the didactical role of discussing that this happens with students (Example of Euler and a false proof quadratic reciprocity). It’s useful sometimes to take a historical perspective to give students the opportunity to realize that mathematicians also make blunders. Mathematicians engage in a process that in some ways can be engaged in by school mathematics learners. For example, it is important to realize that mathematics is coherent and to stay curious, attentive, and active.

Glenn Stevens: History is a wonderful thing to use in a classroom. One appreciates how hard and how significant some ideas are—one has a greater appreciation for how difficult ideas turn into natural ideas.
Paul Goldenberg: Referred to Uri Treisman’s talk in which he discussed balancing urgency with hope. Said in a different way, people are willing to take risks when there is a risk but a good chance. So, this applies to the case of engaging students in mathematical inquiry. Until the student feels that “the game” is winnable, they probably won’t get into it.

Comment (DC?): We’re talking about adding more things to the curriculum for teachers at the same time as we’re having trouble getting people into the professions. We need to think about what it is that could be attractive about teaching to prospective teachers other than giving from their souls.

Cathy Kessel wants Sybilla Beckmann to comment about how mathematics departments try to connect with the NCTM PSSM document. [Not sure if I got the gist of this comment].

Mike Gilbert from U of HI comments on the required course on the historical development of algebra for a masters program (at HI?). He said that engaging in the mathematics helped him to learn more about algebra. He also commented that in terms of developing flexibility and reasoning, it was extremely valuable to have the model of how was mathematics developed by mathematicians throughout history.
Preparing teachers for algebra

MSRI, May 16, 2008

H. Wu
What do algebra teachers need? Let me answer with an anecdote.

Instructor: We now come to a wonderful theorem: if \( f(x) = x^2 + bx + c \) and \( r_1, r_2 \) are the roots of \( f(x) = 0 \), then \( f(x) = (x - r_1)(x - r_2) \) for all numbers \( x \).

Teachers: (silence)

Instructor: Look, it is really wonderful, and let me explain why. It says if I happen to know the two numbers \( r_1, r_2 \) so that \( f(r_1) = f(r_2) = 0 \), then I know everything about the function \( f \). Isn’t it amazing? Who can prove this theorem for me?

One Teacher: (after a long pause) What is there to prove?
Since essentially the same thing happened to me three times in PD, I finally found out what the teachers meant by ”what is there to prove?” Teachers are so used to seeing
\[ f(x) = (x - r_1)(x - r_2) \implies f(r_1) = f(r_2) = 0 \]
that they can no longer see that the preceding theorem is the converse of the theorem they are familiar with.

They conflated a theorem with its converse.

Because school algebra courses are not taught with the requisite precision and rigor, and because universities do not focus on eradicating the common misconceptions of pre-service teachers, anecdotes of this type should be no surprise.
My belief, based on quite extensive experience of this nature, is that to produce teachers with the requisite content knowledge for teaching algebra, we must concentrate on teaching them the fundamentals of mathematics. I am referring to

- proper use of symbols
- precise definitions
- precise reasoning
- coherent development of ideas

There is a particular urgency that they acquire this knowledge, because they have to help students overcome common misconceptions in algebra.
The most basic task of learning algebra, in some sense, is learning how to use symbols fluently and correctly. This is a routine task if one goes about it the right way, but books do not always go about it the right way. The resulting confusion is immense.

**Basic etiquette in the use of symbols:**
*Always say precisely what a symbol stands for.*

Consider:
“$st = ts$ for all real numbers $s$ and $t$.”
In this case, the symbols $s$ and $t$ stand for elements in an infinite collection.

Whenever a symbol stands for elements in a collection of more than one element, we **informally** refer to it as a **variable**. So $s$ and $t$ above are variables.
Consider:
“If $a$, $b$, $c$ are fixed numbers, which number $x$ would satisfy $ax^2 + bx + c = 0$?”
In this case, the symbols $a$, $b$, $c$ stand for a fixed value throughout the discussion. We informally refer to these as **constants**.

The meaning of the *informal* concept of a variable or a constant is thus perfectly simple. *The emphasis should be on knowing exactly what each symbol stands for.* Once that is clear, there would be no confusion about the meaning of such informal labels.
The mathematicians in this audience may be astounded to learn that, in school mathematics, “variable” has achieved the status of a mathematical concept crucial to the study of algebra. Thus,

A variable is a quantity that changes or varies. You record your data for the variables in a table. Another way to display your data is in a coordinate graph. A coordinate graph is a way to show the relationship between two variables.

Sometimes the relationship between two variables can be described with a simple rule. Such rules are very helpful in making predictions for values that are not included in a table or graph of a set of data.
Or,

*Variable* is a letter or other symbol that can be replaced by any number (or other object) from some set. A *sentence* in algebra is a grammatically correct set of numbers, variables, or operations that contains a verb. Any sentence using the verb $=$ (is equal to) is called an *equation*.

A sentence with a variable is called an *open sentence*. The sentence $m = \frac{s}{5}$ is an open sentence with two variables. It is called “open” because its truth cannot be determined until the variables are replaced by values. A *solution* to an open sentence is a replacement for the variable that makes the statement true.
In mathematics, we strive for simplicity whether in teaching or in professional communications. These two examples (among many) subvert this simplicity by **formally defining** an informal piece of terminology in abstruse language. We have to teach teachers how to circumvent these difficulties in the school classroom.

Moreover, the second example encourages the improper use of symbols by formalizing the concept of an “open sentence”. This appears to be not uncommon. In the education literature, one finds similar examples of asking students for interpretations of such “open sentences”, e.g., \( \sqrt{6x - 5} \).

Again, we have to make sure that teachers know well enough not to engage in such counterproductive practices.
Definitions are generally conspicuous by their absence in school mathematics, but because algebra is the gateway course to higher mathematics, this absence is no longer excusable.

Absence here means it is never used in reasoning though it may be given. For example: graph of an equation. This is the reason why there is almost never any proof that the solution of a $2 \times 2$ linear system is the point of intersection of the lines. Students learn by rote that such is the case.

Students come to universities with little respect for definitions. The role of a definition in mathematics is generally not understood in the math education community. Both situations need urgent correction.
There are situations where school texts make it impossible to know whether something is a definition or a theorem, e.g.,

\[(a^{1/n})^n = 1,\]

or,

\[a^{-n} = \frac{1}{a^n}.\]

Consequently, many of our teachers cannot distinguish between a theorem and a definition.

Other basic definitions that are usually MIA: graph of a linear inequality, half-plane, the equivalence of expressions, polynomial form, exponential functions \(a^x\), constant rate.
We now come to **precise reasoning**. Here the “precision” has to be appropriate to the grade level.

Like definitions, reasoning is missing in the algebra curriculum much too often for comfort. Here are some examples:

- why the graph of a linear equation is a line
- why every line is the graph a linear equation
- why the graph of a linear inequality is a half-plane
- why a quadratic function has a max or min, and why the max or min is what it is
- why study the exponential and log functions

**what underlies proportional reasoning**

The math education literature gets **PR** (proportional reasoning) backwards.
The flaw in the teaching of PR is captured by the following problems:

(1) John’s grandfather enjoys knitting. He can knit a scarf 30 inches in 10 hours. He always knits for 2 hours each day.
   a. How many inches can he knit in 1 hour?
   b. How many days will it take Grandpa to knit a scarf 30 inches long?
   c. How many inches long will the scarf be at the end of 2 days? Explain how you figured it out.
   d. How many hours will it take Grandpa to knit a scarf 27 inches long? Explain your reasoning.

(2) On a certain map, the scale indicates that 3 centimeters represents an actual distance of 10 miles. What length does 8.5 centimeters on the map really represent?
There can be no reasoning if the underlying assumption in each problem is not made explicit: grandpa knits at a constant rate, and a **scale map** is a similarity between the actual location and the map.

Of course, we have to teach student the precise definition of “constant rate” and ‘similarity’ in the first place. Definitions are important.

If constant rate has been clearly defined for students, then they would know that the function $f(t)$, which is the number of inches Grandpa knits in $t$ hours ($t \in \mathbb{R}$), is given by $f(t) = ct$ for some constant $c$ (he begins knitting at $t = 0$). *The problem then becomes rather trivial.*
PR is about computations with a *given* linear function without constant terms. A problem about PR can also be explicitly posed as a modeling problem, one that investigates which situations can be modeled by linear functions without constant terms. **In that event, the problems should not be posed the way they are in school textbooks.**

Right now, PR is taught as a guessing game: can you guess that a certain function is actually a linear function without constant term?

This is an interesting activity, but it is **not** mathematical reasoning. It also damages students' learning as it promotes the mindset that the whole world is linear.
Finally, why do we want teachers to have a coherent conception of the development of mathematical ideas? I will once again answer a question with an anecdote:

Q: If a student comes to you and asks why, if \( a \neq 1 \) and \( a > 0 \), \( a^t = a^s \) implies \( t = s \), what would you tell her?

A: \( \log_a a^t = \log_a a^s \), so \( s = t \).

While the answer is 100% correct, mathematically, it is 90% certain that it is all wrong pedagogically. To answer this correctly, the teacher would need instant recall of the whole development that leads up to the definition of \( \log_a x \) and make an educated guess as to where the student’s difficulty may lie, and address that difficulty first.
We want teachers to know, for instance:

(1) the quadratic formula is not just “another formula”, but the high point of a process that yields every desirable conclusion about quadratic functions or equations
(2) the subject of rational expressions is to polynomials as fractions are to whole numbers, and that a knowledge of fractions is a prerequisite for studying rational expressions
(3) the study of linear equation and straight lines depends on congruence and similarity
(4) the factor theorem \((f(r) = 0 \Rightarrow (x-r)|f(x))\) is intimately related to the long division of whole numbers
(5) the precise definition of constant rate simplifies all discussions of rate problems in school mathematics.

This kind of knowledge facilitates teaching.
Fractions, decimals, and rational numbers

http://math.berkeley.edu/~wu/
Remarks on Question 3
Hung-Hsi Wu, UC Berkeley

Wu directs us to his website http://math.berkeley.edu/~wu to a 40 page discussion of the overall architecture of teaching fractions in elementary school which expands on what he has been able to cover in the course of the conference.

Preparing teachers for algebra

Slides One and Two: What do algebra teacher’s need? Wu introduces the talk with an anecdote about teachers confusing the statement for its converse. In the context of school algebra that is not taught with precision and rigor and universities that do not eradicate misconceptions of pre-service teachers, the anecdote from slide 1 isn’t surprising.

Algebra is the introduction to higher mathematics and so teachers have to prepare their students for job. So teachers have to know enough to help their students be ready for higher mathematics.

Slide Three: In PD we need more concentrated efforts over deeper things like:

- proper use of symbols
- precise definitions [know them]
- precise reasoning
- coherent development of ideas

There is a particular urgency so teachers can help students overcome misconceptions

Slide Four: Wu’s take: The most basic task of learning algebra in some sense is to learn how to use symbols fluently and correctly. The basic etiquette in use of symbols is to always say precisely what a symbol stands for. This is where the emphasis should be: on knowing exactly what each symbol stands for.

Slide Five: Whenever a symbols stands for elements in a collection of more than one element, we informally refer it to a variable. The thing that is important is that you know you’re using a symbol for this purpose. Sometimes symbols come in slightly different form: parameters/constants. The term is not important. What’s important is that you’re always aware of how you’re using the symbols.

Slide Six, Seven: Variables and constants are central focus of discussion. The mathematicians in the audience may be astounded to learn that “variable” has become a “mathematical concept.” There are some examples of definitions of variable given from various books in terms of “sentences” and “open sentences.”

Slight Eight: Wu feels that it is not mathematically productive to define informal terminology in abstruse terms. He argues that we need to make sure that teachers do not engage in such counterproductive practices.
Hung-Hsi Wu: Remarks on preparing teachers to teach algebra  
Session 3.3 Discussion around Question 3

Slide Nine: Unfortunately, just because a “definition” appears in a math textbook doesn’t mean that it is being there for the purpose of reasoning with it. Students need to have more respect for definitions and the math education community needs to understand the important role of definitions.

Slide Ten: Deciding whether something is a theorem or a definition? If a characterizing statement is even given, is it a theorem or a definition? Teachers must know the difference between definition and theorem, but many have difficulty doing so. This is probably related to curriculum they have experience with.

Slide Eleven: But not only definitions, but also the reasoning that definitions make possible is missing from the algebra curriculum. Here are some examples:

- Why the graph of a linear equation is a line
- Why every line is a graph of a linear equation
- Why they graph of a linear inequality is a half-plane
- Why a quadratic function has a max or min and why the max or min is what it is
- Why study exponential and log functions
- What underlies proportional reasoning

Slides Twelve and Thirteen: Two problems involving proportional reasoning which capture the “flaw” in the current approach to teaching proportional reasoning.

John’s grandfather enjoys knitting. He can knit a scarf 30 inches in 10 hours. He always knits for 2 hours each day.

- How any inches can he knit in 1 hour?
- How many days will it take Grandpa to knit a scarf 30 inches long?
- How many inches long will the scarf be at the end of 2 days? Explain how you figured it out.
- How many hours will it take Grandpa to knit a scarf 27 inches long? Explain your reasoning.

On a certain map, the scale indicates that 3 centimeters represents an actual distance of 10 miles. What length does 8.5 centimeters on the map really represent?

Wu remarks that certain extremely important assumptions are left tacit in the problem statements. In the first problem, grandpa is meant to be knitting at a constant rate and a scale map is a similarity between the actual location and the map. In addition, Wu argues for the precise definition of “constant rate” and “similarity” to be taught to students.

Slide Fourteen: Proportional reasoning is about computations with a given linear function without constant term. Problems about PR can also be posed as modeling problems. Wu argues that proportional reasoning is often taught as a guessing game: can you guess that a certain function is actually a linear function without constant term? Wu remarks that this approach may have negative consequences in reinforcing an idea that
the whole world is linear.

Slide Fifteen: How would one in a pedagogically sound way address the question of a student who asks why $a^t = a^s$ imply $t = s$ if $a$ is not 1 and $a > 0$. The teacher should have “instant recall” of the whole development leading up to the definition of log base a.

Slide 16: We want teachers to know

- quad form is not just some formula!!!
- Study of rational expressions is to polynomials as fractions are to whole numbers (problems are likely rooted in fraction knowledge)
- Study of linear equations depends on congruence and similarity
- “Factor theorem” is intimately related to long division of whole numbers
- Precise definition of constant rate simplifies discussion of rate problems in school math

The above kinds of knowledge simplify teaching.