① MSRI Monday 06/09/2008

9:00 am

The History of the Weinstein Conjecture I

Celestial Mechanics $\longrightarrow$ Mathematics

(Newton, ...) ~ (1640 – 1710)

(Lagrange) ~ (Hamiltonian systems) ~ (1730–181

(Poincaré) $\to$ (symplectic geometry / dynamical systems)

(1854 – 1912)

Weinstein Conjecture: $\dim(S) = 2n+1$
(1978)

S is a closed ~~odd-dimensional~~ manifold with contact form τ, $X = X_\tau$ is a Reeb vector field

$\tau \wedge d\tau^n \neq 0$

$i_X \tau = 1, \; i_X d\tau = 0$

$\dot{z} = X(z)$ has a periodic point

Hamiltonian Systems:

It was "well-understood" up to the 70's.

1) It is extremely difficult to show the existence of periodic orbits

2) It is very difficult to find closed regular energy surfaces without periodic orbits

② (M, ω) - symplectic manifold

$H: M \longrightarrow \mathbb{R}$ - Hamiltonian

X_H - Hamiltonian vector field

$dH = \omega(X_H, \cdot)$

Consider $\begin{cases} \dot{z} = X_H(z) \\ H(z) = E \in \mathbb{R} \end{cases}$

Explanation of z):

Seifert (1950) observed that it was unknown if every continuous nowhere vanishing vector field on S^3 has a periodic orbit

Wilson (1966): There are vector fields on spheres of odd dimensions (≥ 5) without periodic orbits.

Schweitzer (1974): $\exists$ C^1-vector field on S^3 without periodic orbits

We conclude: For existence theory of periodic orbits we need more structure on the vector field.

K. Kuperberg (1994): There exists a real analytic vector field on S^3 without periodic orbits.

G Kuperberg (1994): C^1-volume preserving vector field without periodic orbits

V. Ginzburg + M. Herman
In $\mathbb{R}^{2n}$, $n \geq 3$ $\exists$ sphere like energy

③ surface without periodic orbits
$\dot{z} = X_H(z), \; H(z) = E$

Existence Theory and Point 1):

Some remarks:

$\underbrace{H, K}_{\hookrightarrow \text{Hamiltonians}} : M \longrightarrow \mathbb{R}$

$\{H = E\} \quad \{K = F\}$

same energy surface S
regular for H and K.

$X_H = f \, X_K$ on S, where $f : S \longrightarrow \mathbb{R} \setminus \{0\}$

Finding periodic points for X_H and X_K on S are the same.

Existence problem for periodic orbits only depends on S

$\mathcal{L}_\omega(S) = \{ (x, h) \in TS \mid \omega(h, h) = 0 \quad \forall h \in T_x S \}$

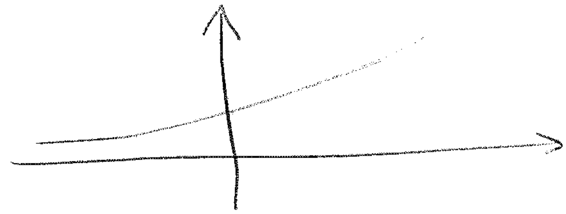
$X_H(S)$ is a section

Background: What did people know about existence of periodic orbits?

— Morse theory for geodesics
first, Morse theory for finite dimensions
then, was generalized to ∞-dimension (1966)

④

$f : \mathbb{R} \longrightarrow \mathbb{R}$



if $\exists$ some compactness condition, namely

$f(x_n) \longrightarrow d,\ f'(x_n) \longrightarrow 0 \Longrightarrow \exists\, x_{n_k}$
↳ convergent subsequence

PS - condition

If PS holds $\Rightarrow$ taking inf $\exists$ critical point

Consider

$z : S^1 \longrightarrow \mathbb{R}^{2n}$; $z = (q_1, p_1, q_2, p_2, \text{—})$

$\lambda = \sum_i dp_i \wedge dq_i$

$$\phi(z) = \int_{S^1} z^* \lambda - \int_{S^1} H(z(t))\, dt$$

$d\,\phi(z) = 0$

z is a 1-periodic orbit

1976

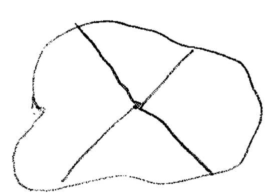
However this variational principle is very degenerate

P. Rabinowitz (1977/1978)

Find periodic orbits for ∞ - dimensional Hamiltonian system

Theorem (Rabinowitz 78)

$\exists$ periodic orbits on star-shaped energy surface in $\mathbb{R}^{2n}$



$S \subset \mathbb{R}^{2n}$