1 06/09/2008 Monday

Seiberg-Witten Equations I

X - closed oriented Riemannian 4-manifold

Choose $Spin^c$-structure s on X

$$(W^+, W^-, \rho) \qquad \rho : T^*X \longrightarrow End(W)$$
$$W = W^+ \oplus W^-$$

$$\rho(\theta)^2 = -|\theta| \mathbb{1}_W, \quad \rho(\theta)^* = -\rho(\theta)$$

Locally i.e., we can choose o.n. basis of T_x^*X $\{e^1, e^2, e^3, e^4$

and $\quad W_x^+$ & $W_x^- \quad \simeq \mathbb{H} = \{x_0 + Ix_1 + Jx_2 + Kx_3$

$$\rho(e^1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(e^2) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$
$$\rho(e^3) = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad \rho(e^4) = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

Equivalent to reducing the structure group of T^*X to $Spin^c \simeq Spin \times_{\mathbb{Z}_2} U$

$Spin^c$-structure are acted on by $\mathbb{C}$-line bundles $L \longrightarrow X$

$$(W^+, W^-, \rho) \longrightarrow (W^+ \otimes L, W^- \otimes L, \tilde{\rho})$$
$$\tilde{\rho} : T^*X \longrightarrow (W \otimes L)^* \otimes (W \otimes L) = W^* \otimes W$$

(2) Iso classes of $spin^c$-structures are principal homogeneous space for iso-classes of $\mathbb{C}$-line bundle $\simeq H^2(X, \mathbb{Z})$

$$\rho(vol) = \rho(e^1)\rho(e^2)\rho(e^3)\rho(e^4) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda^2 = \underset{*=1}{\Lambda^+} \oplus \underset{*=-1}{\Lambda^-} \qquad \begin{pmatrix} e^1\wedge e^2 + e^3\wedge e^4 \\ e^1\wedge e^3 - e^2\wedge e^4 \\ e^1\wedge e^4 + e^2\wedge e^3 \end{pmatrix}$$

$$\rho(e^1\wedge e^2 + e^3\wedge e^4) = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\rho(\Lambda^+) \subset su(W^+), \quad \rho(\Lambda^-) \subset su(W^-)$$

Make $\mathbb{H}$ into a vector space by left multiplication by I

$$|e^1\wedge e^2 + e^3\wedge e^4|^2_{\Lambda^+} = 2$$

$$|\rho(e^1\wedge e^2 + e^3\wedge e^4)|^2_{su(W^+)} = 4 \qquad |A|^2 = -\frac{1}{2}\,tr(A^2)$$

Given an almost complex structure $J: T^*X$

$J^2 = -1$

J metric compatible $J^* = -J$

$\Rightarrow$ get a principal $Spin^c$-structure

③

$$W^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$$

$$W^- = \Lambda^{0,1}$$

$$\rho = \sqrt{2} \circ (\mathcal{J} + \mathcal{J}^*)$$

Clifford Connection

$$\nabla_A : \Gamma(W) \longrightarrow \Gamma(T^*X \otimes W)$$

→ Levi-Civita connection

$$\nabla_A (\rho(\theta)\varphi) = \rho(\overset{LC}{\nabla}\theta)\varphi + \rho(\theta)\nabla_A \varphi$$

The space of Clifford connection is an affine space for $i\Omega^1(X)$.

$$\nabla_A + a\,\mathbb{1}_W$$

$$\nabla_A : \Gamma(W^\pm) \longrightarrow \Gamma(T^*X \otimes W^\pm)$$

A^t = induced connection in $\det(W^+)$

Dirac Operator:

$$D_A : \Gamma(W^+) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes W^+) \xrightarrow{\rho} \Gamma(W^-)$$

$\mathcal{A}_W$ = space of Clifford connection.

$$\mathcal{C}_S = \mathcal{A}_W \times \Gamma(W^+)$$

$$sw : \mathcal{C} \longrightarrow \mathcal{R} = i\mathfrak{su}(W^+) \times \Gamma(W^-)$$

$$sw(A, \varphi) = \Big(\overbrace{\tfrac{1}{2}\rho(F_{A^t}^+) - (\varphi\varphi^*)_0}, \overbrace{D_A\varphi}\Big)$$

④ $\phi\,\phi^* \in \mathrm{iu}(W^+)$

$$(\phi\,\phi^*)_0 = \phi\,\phi^* - \frac{1}{2}|\phi|^2\,\mathbb{1}_{W^+}$$

Energy Identity

$$\int_X |SW(A,\phi)|^2 = \frac{1}{4}\int |F_{A^t}|^2 + \int |\nabla_A \phi|^2 +$$

$$+\frac{1}{4}\int \left(|\phi|^2 + \frac{s}{2}\right)^2 - \frac{1}{16}\int s^2 + \pi^2 \langle c_1^2(W^+), [X]\rangle$$

scalar curvature

$$L_1^2 \longrightarrow L^4$$

(in the case of J-holo curves it is analogous for energy formula)

(energy of map + topological term)

$$\int_\Sigma |\bar\partial u|^2 = \int_\Sigma u^* w + \int_\Sigma |du|^2$$

$$\int |D_A\phi|^2 = \int |\nabla_A \phi|^2 + \frac{s}{4}|\phi|^2 + \frac{1}{2}\langle \cancel{\rho(F_A^+)\phi, \phi}\rangle$$

Weitzenböck formula

$$\int \left|\frac{1}{2}\rho(F_A^+) - (\phi\,\phi^*)_0\right|^2 = \frac{1}{2}\int |F_A^+|^2 -$$

$$-\int \langle \cancel{\rho(F_A^+), (\phi\,\phi^*)_0}\rangle + \frac{1}{4}\int |\phi|^4.$$

(do adding)

Symmetries: $\mathcal{G} = \mathrm{Map}(X, S^1)$ act on $\mathcal{C}$, R

$$\mathcal{C}\begin{cases} u: A \longrightarrow A - u^{-1}du\,\mathbb{1}_W \\ u: \phi \longrightarrow u\phi \\ u: \omega \longrightarrow \omega \quad \mathrm{isu}(W^+) \end{cases} \quad u: \psi \longrightarrow u\psi \quad W^-$$

(5) $SW(u(A,\phi)) = u\, SW(A,\phi)$.

$SW^{-1}(0)/\mathcal{G} = M$ the modular space of solutions to SWE.

Understand $\mathcal{C}/\mathcal{G}$

Fix $A_0 \in \mathcal{C}$ there is an almost slice

$$\mathcal{A}_{A_0} = \{ (A_0 + a, \phi \mid d^* a = 0 \}$$

$\forall (A \varphi)$ $\exists !$ $u = e^{if}$, $f \in C^\infty(X, \mathbb{R})$

$\int f = 0$

so $u \cdot (A, \varphi) \in \mathcal{A}_{A_0}$

$\left(\text{Solve } \begin{array}{l} d^*(u^{-1}du) = d^* a \\ i \Delta f = d^* a \end{array} \right)$

If $(A, \phi) \in \mathcal{A}_{A_0}$

$u \cdot (A, \phi) \in \mathcal{A}_{A_0} \Rightarrow d^*(u^{-1}du) = 0$

i.e. $u : X \longrightarrow S^1$ is a harmonic map.

$\mathcal{C}/\mathcal{G} = \mathcal{A}_{A_0} / \mathcal{G}^A$ $\quad U_1 \longrightarrow \mathcal{G}^* \longrightarrow H^1(M, \mathbb{Z})$

= coclosed form / harmonic form with integral powed $\times \Gamma(W^+)/U_1$

$\simeq$ " $T^{b^1(X)} \times \mathbb{R}^\infty \times C(\mathbb{C}P^\infty)$.

Assume: $b^1(X) = 0$

⑥ $sw : \mathcal{A}_{A_0} \longrightarrow R$

Claim: This is a proper Fredholm map

$ds\,w(a, \varphi) = (\rho(d^+a) - (\underbrace{\phi\varphi^* + \varphi\phi^*)_0}, \not{g}^h)$

$D_A \varphi - \underbrace{\rho(a)\phi}$

cpt operator is Fredholm

Since $\{a \mid d^*a\} \longrightarrow d^+a \in \Gamma(\Lambda^+)$

is Fredholm (index $b^1(X) - b^+(X)$)

and D_A is Fredholm index $\frac{1}{4}(\langle c_1^2(W^+), [X]\rangle + \sigma(X))$

$\text{Index}(d_{(A,\phi)} sw) = \frac{1}{4}(\langle c_1^2(W^+), [X]\rangle + 2\chi + 3\sigma) +$

$+ \underset{\text{C from } U'C\ \not{g}^h}{+}$

Proper: $sw : \mathcal{A}_{A_0}^{L_1^2} \longrightarrow R^{L^2}$

$\{(A_i, \phi_i)\} \in \mathcal{A}_{A_0}^{L_1^2}$ so that $sw(A_i, \phi_i)$ converges in L^2

Energy identity $\Rightarrow \|F_{A_i^\tau}\|_{L^2}^2, \|\nabla_A \phi_i\|^2, \|\phi_i\|_{L^4}^4 \leq M$

can pass to subsequence where

$A_i \to L_1^2 \Leftarrow F_{A_i^\tau} \xrightarrow{L^2} \nabla_{A_i} \xrightarrow{L^2} \phi_i \xrightarrow{L^4} \phi$

⑦ $(\varphi_i, \varphi_i^*)_0 \xrightarrow{L^2} (\phi\, \varphi^*)$

$$\begin{array}{ccccc} \nabla_{A_i} \varphi_i & = & \nabla_{A_0} \varphi_i & + & a_i \varphi_i \\ \downarrow & & \downarrow & & \downarrow \\ \nabla_A \phi & = & \nabla_{A_0} \phi & + & a\phi \end{array}$$

In the weak limit, no loss of norm the Energy Identity $\Rightarrow$ All limits are strong

Take a regular value (w, ϕ)

$sw^{-1}(w, \underset{\psi}{\phi}) / \mathcal{G}^h$ (need $\psi = 0$ for action)

In fact $\mathcal{F} \subset \mathcal{A}_{A_0} = \{(A, \phi) \mid D_A \varphi = 0\}$ is a smooth manifold away from $\varphi = 0$.

$sw|_{\mathcal{F}} \longrightarrow L^2(X, i\,su(A^{\tau}))$ is proper Fredholm

Take regular value $sw^{-1}(w, \theta)/\mathcal{G}h$ is a smooth "cpt" manifold of dimension $\frac{1}{4}(c_1^2(w^+) - 2\chi(X) + \sigma(X))$ (Away from $\phi = 0$).

If $b^+(X) > 1$ then for generic w and generic 1-parametric family between generic w's

(8) $SW^{-1}(W,0)/\mathcal{G}$ is a cpt manifold cobordant through compact manifold

E.g.: if formal dim $= 0$, $b^+(X) > 1$

count of solution mod 2 is an invariant (indeed in $\mathbb{Z}$ with sign)

$X = \mathbb{R} \times Y$, $A = A_0 + b + c\,dt^{=0}$

$SW(A, \phi) = 0$; $b'^{\tau} = -(*db^{\tau} + \rho^{-1}((\phi\phi^*)_0))$

$\phi' = -D_B^{Y}\phi$

$$CSD(b, \phi) = -\frac{1}{8}\int_Y b \wedge db + \frac{1}{2}\int_Y (\phi, D_B^{Y}\phi)$$

$$B = A_0 + b|_Y$$

This is the downwards gradient flow for CSD functional