① - Richard Siefring 06/09/2008

The Asymptotic Behavior of punctured Pseudoholomorphic curves

Motivation: ECH-index.

$u : \Sigma \setminus \Gamma \longrightarrow \mathbb{R} \times M$

$I(u) \geq \operatorname{ind}(u)$

"=" $\Longleftrightarrow$ u is an embedding + some tech. conditions

Background:

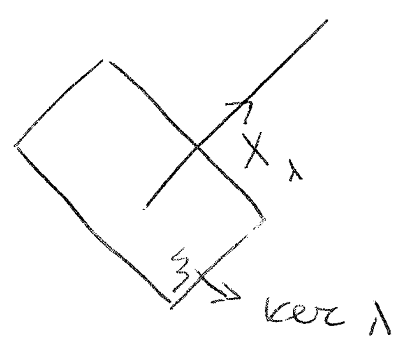
Stable Hamiltonian structure

(M^3, λ, ω)

λ - 1-form, ω - 2-form

1) $\lambda \wedge \omega$ - volume form
2) $d\omega = 0$
3) $d\lambda$ vanishes on kernel of ω.
(namely on kernel of $v \longmapsto i_v(\omega)$)

Flow of X_λ preserves ξ



$\lambda(X_\lambda) = 1$

(2) Define a class of almost-complex structures

Choose $J \in \mathrm{End}(\xi)$; $J^2 = -I$

compatible with metric $\omega(\cdot, J\cdot)$

↳ metric on $\xi \times \xi$

Extend J by $\tilde{J}\partial_a = X_\lambda$, where a is $\mathbb{R}$-coordinate on $\mathbb{R} \times M$.

$\gamma : S^1 = \mathbb{R}/\mathbb{Z} \longrightarrow M$

$\dot{\gamma} = T \cdot X_H$

↳ period of orbit

We want to study:

$u : [0, \infty) \times S^1 \longrightarrow \mathbb{R} \times M$:

$du \circ j = \tilde{J} \circ du$

which converge to cylinder over periodic orbits.

Example: $M = S^1 \times \mathbb{R}^2$

ϕ-parameter, (r, θ)

$\lambda = d\phi$ ↳ polar

$\omega_\alpha = r\, 2\pi\alpha\, d\phi \wedge dr + \omega_0$

$\lambda = d\phi \qquad dx \wedge dy = r\, dr \wedge d\theta$

$X_\lambda = \partial_\phi + 2\pi\alpha\, \partial_\theta$

$\xi = T\mathbb{R}^2$

Choose J on ξ to be $J\partial_x = \partial_y$

③ $\tilde{J} = \begin{bmatrix} i & 0 \\ \Delta(x,y) & i \end{bmatrix}$ $\quad T(\mathbb{R} \times S^1 \times \mathbb{R}^2)$

$$\Delta(x,y) = 2\pi\lambda \begin{bmatrix} -y & x \\ x & y \end{bmatrix}$$

Look at pseudoholomorphic maps of the form

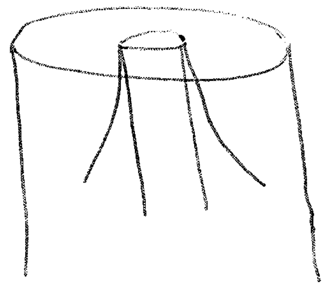
$(s,t) \in \mathbb{R}^+ \times S^1 \longrightarrow (s, t, h(s,t))$, where

h satisfies $\quad h_s + i h_t + 2\pi\lambda h = 0$.

$$h = \underbrace{\sum_{k \in \mathbb{Z}}}_{k \ll \lambda} c_k \, e^{2\pi(k-\lambda)s} e^{i 2\pi k t}$$

↖ Fourier series for h.

H-W-Z (1996) + Mora (2003



$$u \circ \phi(s,t) = (\tau s, \exp_{\gamma(t)} u(s,t))$$

(τ: partial; ϕ: reparametrization)

$u(s,t) \in \xi_{\gamma(t)}$

$u(s,t) = e^{\lambda s}[e(t) + r(s,t)]$, where $\lambda < a$ is eigenvalue of $A_{J,\gamma} = -J(\nabla_t - \tau \nabla_\cdot X_H)$.

(4) e is an eigenvector with eigenvalue λ, and $D_i D_j \; r(s,t) \longrightarrow 0$ exponentially in s for all (i,j)

Pick a unitary trivialization $-i\frac{d}{dt} + S(t)$

$$(\tau s, \exp_{\gamma(t)} u(s,t))$$

$$(\tau s, \exp_{\gamma(t)} v(s,t))$$

$$u = e^{\lambda_1 s}(e_1(t) + r_1(s,t))$$

$$v = e^{\lambda_2 s}(e_2(t) + r_2(s,t))$$

$$u(s,t) - v(s,t) = e^{\lambda_i s}(e_i(t) + r(s,t))$$

Theorem (for this theorem)

$$u(s,t) - v(s,t) = e^{\lambda s}(e(t) + r(s,t))$$

$\lambda < 0$ - eigenvalue of A_t, γ

$e \neq 0$ - eigenvector with eigenvalue λ

$r \to 0$ on $s \to 0$ exponentially

$$(\tau s, \exp_{\gamma^k_\ell(t)} u(s,t))$$

$$u(s,t) = \sum_{j=1}^{\ell} e^{\lambda_j s}(e_j(t) + r_j(s,t))$$

where λ_j is decreasing sequence of negative eigenvalues

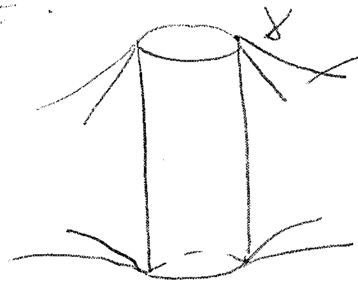
e_j - eigenvectors

$$K_1 = \operatorname{cov}(e_1), \; K_i = \gcd(K_{i-1}, \operatorname{cov}(e_i))$$

strictly decreasing

$$r_j\left(s, t + \frac{1}{K_j}\right) = r_j(s,t)$$

⑤ Result:



probably with different covering #'s

Can choose coordinates near $\mathbb{R}\times\gamma$ in $\mathbb{R}\times S\times\mathbb{R}^2$ so that every pseudoholomorphic half-cylinder is of the form

$$(s,t)\longmapsto \left(Ks,\ Kt,\ \sum{}^{\lambda_i s} e_i(t)\right)$$

$$\partial_s - A_{\tau,\gamma} = 0$$

Proof: (of theorems)

If $h: \mathbb{R}\times S \longrightarrow \mathbb{R}^{2n}$:

$$\partial_s h + J\partial_t h + S(t)\,h + \sigma(s,t)\,h = 0 \implies$$

0 exponentially

al. compl. structure

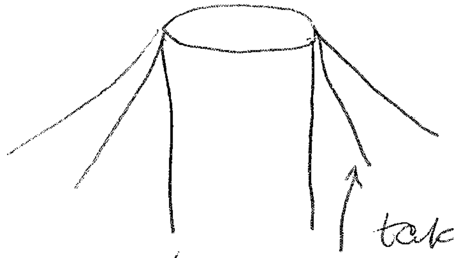
loop of symmetric matrices

$$\implies h(s,t) = e^{\lambda s}\left(e(t) + v(s,t)\right) \qquad -J\frac{d}{dt} - S(t)$$

Fact that is used for proof

The linearization of C-R equation at an orbit cylinder is

$$\nabla_s - A_{J,\gamma}$$



take one curve and use it for another curve (not ...)

$$(s,t) \longmapsto \Big(s, t, \sum_{k<\alpha} e_k\, e^{2\pi(k-\alpha)s} e^{i2\pi k t}\Big) \in \mathbb{R}\times S^1 \times \mathbb{R}$$

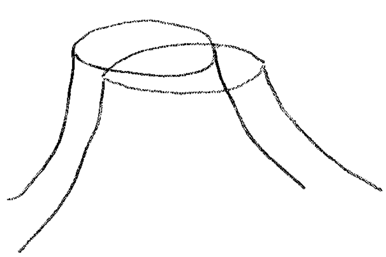
$$u(s,t) = \left(s, t\, e^{-2\pi(1+\alpha)} e^{-i\alpha\pi t}\right)$$

$$v_\varepsilon(s,t) = \left(s, t, e^{-2\pi(1+\alpha)s}(1+\varepsilon)e^{-2\pi i t} + e^{-2\pi(2+\alpha)s} e^{-i4\pi t}\right)$$

$$u(s,t) - v_\varepsilon(s,t) = -\varepsilon\, e^{-2\pi(1+\alpha)s} e^{-2\pi i t} - e^{-2\pi(2+\alpha)s} e^{-4i\pi t}$$

For $\varepsilon \neq 0$ these two curves intersect once
For $\varepsilon = 0$ intersection is 0
(tangency at ∞).

$$\left(\tilde{s}, \exp_{\gamma(t)} u(s,t)\right) \qquad \left(\tilde{s}, \exp_{\gamma(t)} v(s,t) + \phi^{-1}(\varepsilon)\right)$$



$$\underbrace{u(s,t) - v(s,t)}_{} + \phi^{-1}(\varepsilon)$$

for small s

$$e^{\lambda s}\left(e(t) + r(s,t)\right)$$

$$-\mathrm{wind}\, \phi^{-1} e > -\left[\frac{\mu\phi(\gamma)}{2}\right]$$