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Introduction to the S-W-F homologies I

$(B, \partial B)$ → finite-dimensional compact manifold

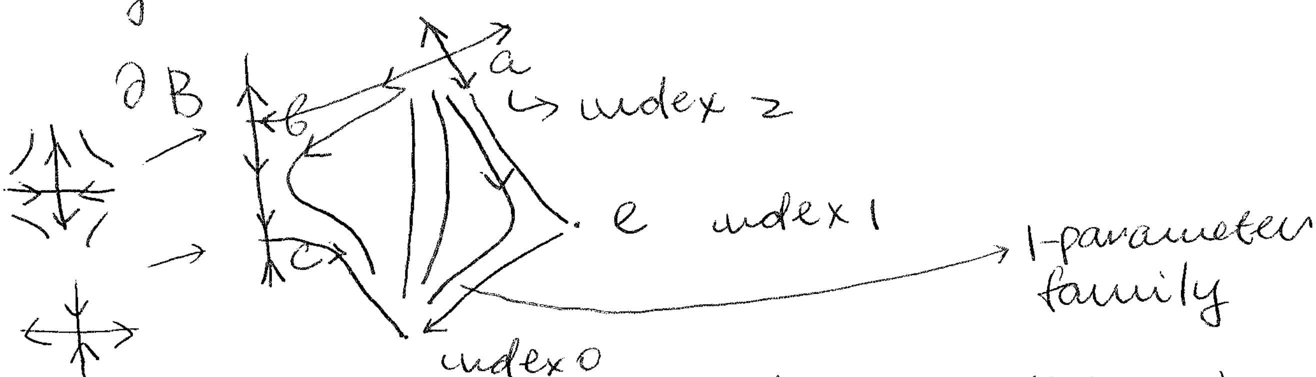
$\partial B \neq \emptyset$

$f : B \longrightarrow \mathbb{R}$ - Morse function

$V = -\operatorname{grad}(f)$

Suppose f is tangent to ∂B.

Eg : dim 2



b, c - critical points $(V(b) = 0,\ V(c) = 0)$

index 1

$M(b, c) = \overset{\text{unstable}}{U_b} \cap \overset{\text{stable}}{S_c} \neq \emptyset$

$T_b S_b$ contains $(\partial B)^{\perp}$

$T_c U_c$ contains $(\partial B)^{\perp}$

We say that b is "boundary-stable"

——, —— c is "boundary-unstable"

② $\mathrm{ind}(b) = \mathrm{ind}_\partial(b)$

↳ boundary index

$\mathrm{ind}(c) = \mathrm{ind}_\partial(c) + 1$

The flow line is „boundary-obstructed"

Form chain complex:

$$C_*(f) = \bigoplus_{a \in \mathrm{Crit}(f)} \mathbb{F}$$

← we do it like that (in this way) in the closed case

$\mathbb{F} = \mathbb{F}_2 = \mathbb{Z}/2$

∂ count isolated trajectories in $M(a,a)/\mathbb{R}$

$$C_* = C^o_* \oplus C^s_* \oplus C^u_*$$

interior „irreducible" — boundary stable — boundary unstable

$$\bar{C}_* = C^s_* \oplus C^u_*$$

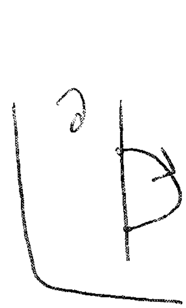
$$\bar{\partial} = \begin{bmatrix} \bar{\partial}^s_s & \bar{\partial}^u_s \\ \bar{\partial}^s_u & \bar{\partial}^u_u \end{bmatrix}$$

$\bar{H}_*(f) \cong H_*(\partial B)$

Compute $H_*(B)$

$$\check{C}_* = C^o_* \oplus C^s_*$$

$$\check{\partial} = \begin{bmatrix} \partial^o_o & \partial^u_o \bar{\partial}^s_u \\ \partial^o_s & \bar{\partial}^s_s + \partial^u_s \bar{\partial}^s_u \end{bmatrix}$$



$\bar{\partial}$ counts on the boundary

③ Proposition: $\check{H}_*(f) := H(\check{C}_*, \check{\partial}) \cong H_*(B)$

$$\hat{C}_* = C^o_* \oplus C^u_*$$

$$\hat{\partial} = \begin{bmatrix} \partial^o_o & \partial^u_o \\ \bar{\partial}^s_u \partial^o_s & \bar{\partial}^u_u + \bar{\partial}^s_u \partial^u_s \end{bmatrix}$$

Proposition: $\hat{H}_*(f) := H(\hat{C}_*, \hat{\partial}) \cong H_*(B, \partial B)$

There is a LES relating $\bar{H}$, $\check{H}$ and $\hat{H}$:

$$\cdots \longrightarrow \bar{H}_* \longrightarrow \check{H}_* \longrightarrow \hat{H}_* \longrightarrow \bar{H}_* \longrightarrow \cdots$$

II Linear flows

W - complex v space

$\mathbb{C}P^N = \mathbb{P}(W)$

$L: W \longrightarrow W$

↳ self-adjoint

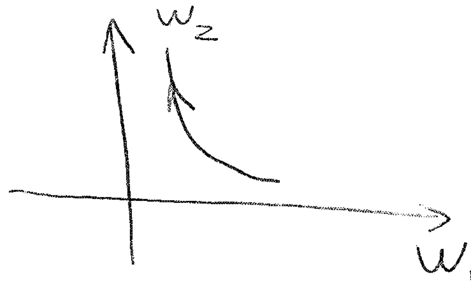
On W: $\dot{w} + Lw = 0$

$f = \frac{1}{2}\langle w, Lw \rangle$

On $\mathbb{P}(W)$: $\Lambda([w]) = \dfrac{\frac{1}{2}\langle w, Lw \rangle}{\langle w, w \rangle}$

Flow lines in W for f project to flow-lines for Λ in $\mathbb{P}(W)$

(4)



$a_1 = [w_1]$

$a_2 = [w_2]$

$\mathbb{P}(w)$

$\left.\begin{array}{l}\lambda_1 \\ \lambda_0\end{array}\right\} > 0$

$\left.\begin{array}{l}\lambda_{-1} \\ \lambda_{-2}\end{array}\right\} < 0$

W_i e-vectors of L

Spec (L) is simple

$0 \notin$ Spec

$$\operatorname{ind}(a_i, a_{i-1}) = \operatorname{ind}(a_i) - \operatorname{ind}(a_{i-1}) = 2$$

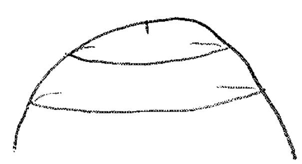
W, suppose $\lambda_0 = 0$

$$g = \frac{1}{2}\langle w, Lw\rangle - |z_0|^4$$

$$g_\varepsilon = \frac{1}{2}\langle w, (L+\varepsilon)w\rangle - |z_0|^4$$

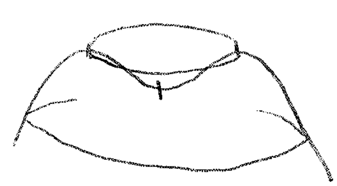
On z_0 coordinate

$\varepsilon < 0$



$\mathbb{C}$ $\quad \varepsilon|z|^2 - |z|^4$

$\varepsilon > 0$

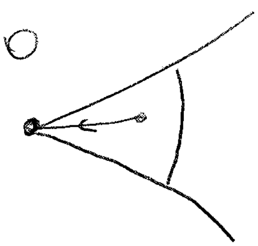


(a) Spectral flow (positive) for $\mathrm{Hess}_\theta(f)$

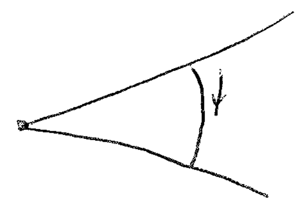
(b) An S^1 at critical points is born

⑤ In $W/S^1 = \mathrm{Cone}(\mathbb{P}(W))$

$\varepsilon > 0$



$\varepsilon < 0$

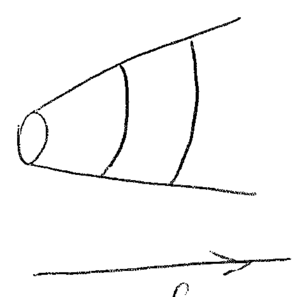


III $\quad \sigma =$ "blow-up"

$W^\sigma = \mathbb{R}^+ \times S(W)$

$B = (W^\sigma)/S^1$

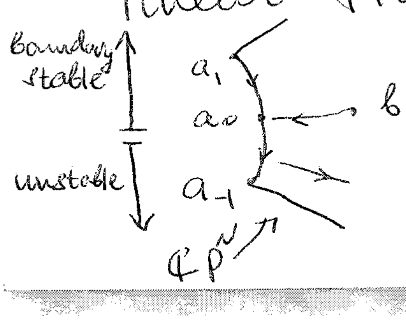
$= (W/S^1)^\sigma$



V defines flow on $(W \setminus 0)/S^1$

V extends to V^σ-flow on $(B, \partial B)$

On $s = 0$ $(\partial B = \mathbb{P}(W))$, it is just a linear flow for Λ $(L = \mathrm{Hess}_0(f))$



V^σ flow, $\varepsilon > 0$

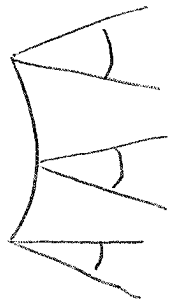
can construct $\check{C}_*$, $\hat{C}_*$, $\bar{C}_*$

Consider P with S^1 action
↳ manifold

$Q \subset P$ fixed by S^1
↳ submanifold

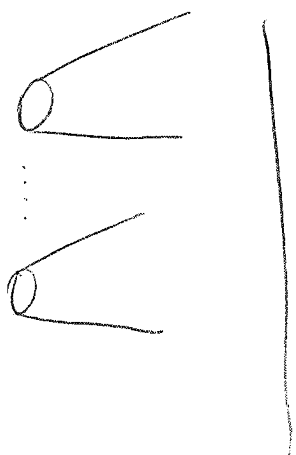
Q free elsewhere

P/S^1



$$B^\sigma = (P/S^1)^\sigma$$

$$\begin{array}{ccc} \partial B^\sigma & \hookleftarrow & \mathbb{C}P^N \\ \downarrow & & \\ Q & & \end{array}$$

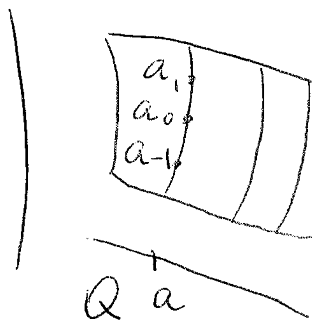


f S^1-invariant on P

Get V^σ on B^σ

Zeros of V^σ on ∂B^σ lie over $a \in Q$ which are critical points of $f|_Q$

Over a, critical points in $\mathbb{C}P^N$ are e-directions of $\mathrm{Hess}_a(f)$ restricted to $Q^\perp$



IV Y^3, $\mathfrak{s}$ - Spin^c-structure

$$S \longrightarrow Y^3 \quad \text{fiber } \mathbb{C}^2$$

$$\mathcal{C} = \mathcal{A} \times \Gamma(S)$$

$\mathcal{A}$ = all spin^c connections =

$$= \{ A_0 + a\,\mathbb{1}_S \mid a \in \Omega^1(i\mathbb{R}) \}$$

$$(A, \Phi) \in \mathcal{A} \times \Gamma(S)$$

Acted on by $\mathcal{G} = \mathrm{Map}(u : Y \longrightarrow S^1)$

$$\mathcal{B} = \mathcal{C} / \mathcal{G}$$

(A, Φ), $\Phi \neq 0$ (irreducible)

Stab in $\mathcal{G}$ is $\{1\}$

$(A, 0)$ (reducible)

Stab is S^1 (u=const)

(8) $\mathcal{C}^{\sigma} = \mathcal{A} \times \Gamma(S)^{\sigma}$

$$= \mathcal{A} \times \mathbb{R}^{+}_{S} \times S(\Gamma(S))$$

$$B^{\sigma} = \mathcal{C}^{\sigma} / \mathcal{C}$$