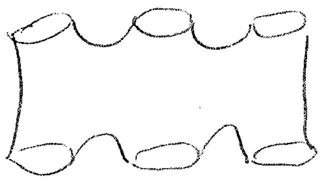
① Ko Honda 06/11/08

Relative Contact Homology

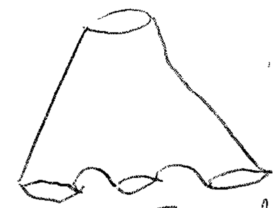
(joint with Colin, Ghiggini, Hutchings)

Goal: Define a version of CH or ECH for 3-manifolds with boundary.

ECH: counts J-holo with $I=1$.



CH:



with Fredholm 1.

Linearized CH: counts J-holo with Fredholm index 1

1. Sutured boundary condition.

Def: A sutured manifold (M,Γ) consists of:

1) M^3 = compact with corners
2) Γ = union of annuli $\subset \partial M$
   ↳ disjoint
3) $\partial\Gamma$ = corners
4) $\partial M \setminus \Gamma = \overbrace{R_+(\Gamma) \vee R_-(\Gamma)}^{\text{alternate}}$

$R_+(\Gamma)$ ↳ or-n agrees with ∂M or-n

$R_-(\Gamma)$ ↳ opposite to ∂M orientation

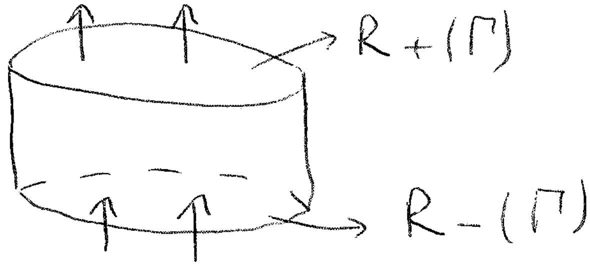
no 0-dim-1 corners

has to be with the boundary (*)

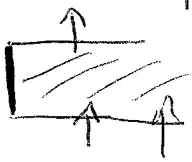
(2) Ex: product sutured manifold

$\partial\Sigma \neq \emptyset$

$(\Sigma \times [0,1],\ \partial\Sigma \times [0,1])$



local picture



(think about $\times S^1$

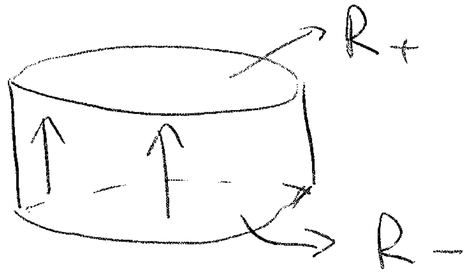
in M

Def: A contact form α / Reeb v.f. R_α is adapted to (M, Γ) if:

(1) $R_\alpha \pitchfork R_+(\Gamma)$ and exits M

(2) — " — $R_-(\Gamma)$ and enters M

(3) R_α is tangent to Γ and on each component $S^1 \times [0,1]$ is of the form $\frac{\partial}{\partial t}$, where t - direction of $[0,1]$



③. (1), (2) $\Rightarrow$ Near $R_\pm(\Gamma)$ $\quad \alpha = dt + \beta$
$\qquad d\beta > 0$ on $R_\pm(\Gamma)$

if (*) is missed $\rightarrow$ will be wrong

(4) (Technical condition)

$\beta = df \circ J$, where:

J = complex structure on $R_\pm(\Gamma)$ which makes it Stein and f is strictly plurisubharmonic on $R_\pm(\Gamma)$

Ex: PSM (product sutured m-d)

$\alpha = dt + \beta, \quad d\beta > 0$

$\nexists$ closed orbits

Thm ECH (M, Γ, α) can be defined in the usual way.

* Ko Honda does not say that ECH(M, Γ, α) independent of α.

$\rightarrow$ actually says:

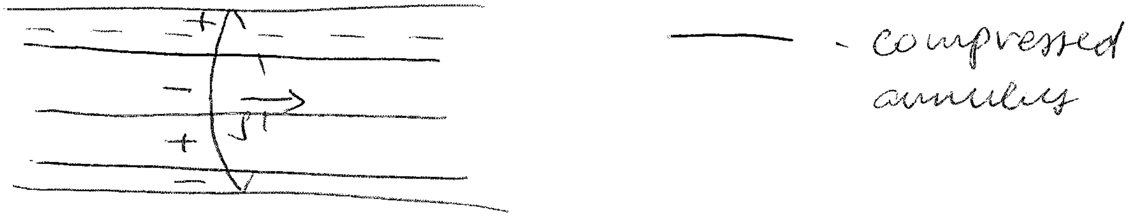
No reference of holomorphic curves between γ and γ' exits the sides / top and bottom

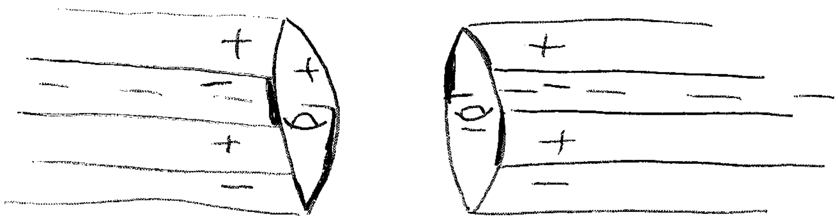
ECH(PSM) $= \mathbb{Z}$

$\hookrightarrow$ because of $\emptyset$

2. Sutured manifold splitting

$$(M,\Gamma) \overset{S \to \text{oriented}}{\rightsquigarrow} (M',\Gamma')$$





Sutured manifold hierarchy

$$(M,\Gamma) \rightsquigarrow (M_1,\Gamma_1) \rightsquigarrow \cdots \rightsquigarrow \underbrace{\coprod (PSM)}_{\text{already constr contact form}}$$

Th (Colin-H)

Given a hierarchy, one can construct (M,Γ,α) which has no contractible periodic orbits

($\Rightarrow \exists$ cylindrical CH)

Conjecture: Closed M, F = genus-minimizing surface $[F] \neq 0 \in H_2(M,\mathbb{Z})$, then

$$\bigoplus_{\langle c_1(\mathfrak{s}),F\rangle = 2-2g} ECH(M,\overset{\text{spin}^c\text{-structure}}{\mathfrak{s}}) = \begin{cases} \mathbb{Z}, & \text{if } M \text{ is fibered} \\ \geq \mathbb{Z}^2 & \text{if } M \text{ is not} \end{cases}$$

(5)

For HF, this is a theorem due to Ni, based on Ghiggini's ideas.

Missing ingredient

Well-definition / invariance of ECH when extended to "stable" Hamiltonian systems.

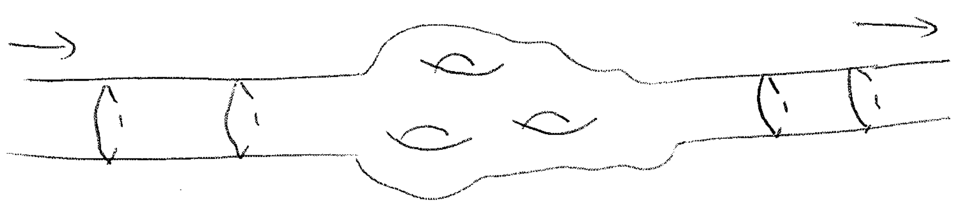
Step 1 $ECH(M,\Gamma,\lambda) = \mathbb{Z}$ if product

Step 2: Claim $(M,\Gamma,\lambda) \geq \mathbb{Z}^2$ if

$(M,\Gamma,\lambda) \rightsquigarrow (M',\Gamma',\lambda')$ if (M',Γ',λ') is not a product.

Two ingredients:

1) Understanding homeos of surfaces which are end-periodic



Handel-Miller theory
(which generalizes Nielsen-Thurston theory)

2) You take Euler characteristics = # even orbits − # odd orbits

described in terms of hyperbolic and elliptic orbits

over fixed Nielsen class → things that are in the same Nielsen class → gives extra $\mathbb{Z}$ cylindrical are

(Work of Cotton-Clay on sympl. Floer homology of surface diffeo.

⑥ Step 3: $(M, \Gamma, \alpha) \rightsquigarrow (M', \Gamma', \alpha') \Rightarrow$

$\Rightarrow \exists$ a direct summand map

$$ECH(M', \Gamma', \alpha') \hookrightarrow ECH(M, \Gamma, \alpha)$$

complex consisting of orbits that do not intersect S

To fell "Hamiltonian problem".
(possible idea)
when we do sutured decomposition we cut link $\Rightarrow$ have toroidal suture and at the end we feel it in

3 Further remarks on ECH.

What is $\widehat{ECH}$?

analogue of $\widehat{HF}$

Algebraic definition:

$$\begin{tikzcd} ECH(M) \arrow[rr, "U"] & & ECH(M) \arrow[dl] \\ & \mathrm{Cone}(U) \arrow[ul] & \end{tikzcd}$$

$\mathrm{Cone}(f: V \to W)$

boundary: $\begin{pmatrix} \partial_V & 0 \\ f & \partial_W \end{pmatrix}$

$$\begin{tikzcd} V \arrow[r, "\partial_V"] \arrow[dr] & V \\ \oplus & \oplus \\ W \arrow[r, "f"', "\partial_W"] & W \end{tikzcd}$$

Geometric definition:

$\widehat{ECH}(M)$

by analogy with sutured Floer homology, due to Juhász

$$SFH(M - B^3, s) \simeq \widehat{HF}(M)$$

(7) Theorem

alg = geom

Knot ECH $\overset{\text{def}}{=}$ ECH $(M - N(K)$, 2 parallel meridian sutures$)$

$\| $

HFK

Conj: $ECH(M, \Gamma) \cong SFH(M, \Gamma)$

Evidence: 1) $ECH(PSM) = \mathbb{Z} = SFH(PSM)$

2) (Golovko)

$ECH(S^1 \times D^2, S^1 \times 2n \text{ points on } \partial D^2) \cong SFH$

$= (\mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(-1)})^{\otimes n-1}$

C. Tauber 06/10/2000