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Existence theorems for SWF homology

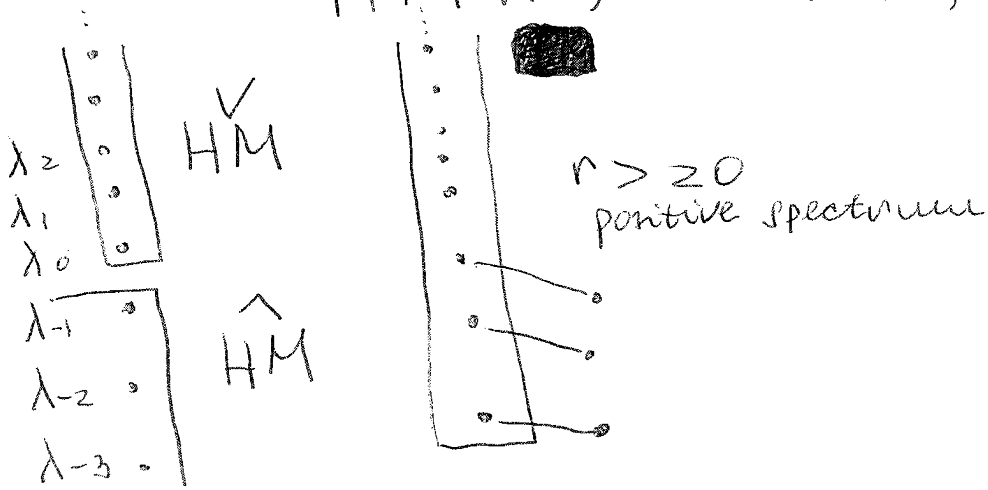
(Y, s) is a 3-manifold with spin^c s

S - spin bundle on Y.

Theorem: If $c_1(s)$ is zero on torsion, then $\widehat{HM}_*(Y, s)$ is non-zero in ∞-ly many negative degrees

$\check{HM}_*(-Y)$

S^3 $\overline{HM}(S^3)$ $\mathbb{Z}[T^{-1}, T]$



Enough to look at $\overline{HM}(Y, s)$

(A module over $\mathbb{Z}[T^{-1}, T]$

Show it has $\neq 0$ rank

Q compact

$K^{-1}(Q) = [Q, U(\infty)]$ $= \lim_{n \to \infty} U(n)$

(2) D_q, $q \in Q$, a family of Dirac operators,

$$D_q : \Gamma(Y, S_q) \longrightarrow \Gamma(Y, S_q)$$

odd-dim $\Rightarrow$ self adjoint

Atyah-Singer

Define $\mathrm{ind}_{top}(D) \in K^{-1}(Q)$

$\mathrm{ind}_{an}(D) \in K^{-1}(Q)$

↳ analytic

Theorem: (A-S)

$\mathrm{ind}_{top} = \mathrm{ind}_{an}$

$\begin{pmatrix} \text{self-adjoint} \\ \text{Dirac op.} \end{pmatrix} \subset (?) \simeq U(\infty)$ operators

unbounded — bounded

[redacted] smoothing

What is (?)

Fix $H_1 \subset H$ (Hilbert)

e.g. $L_1^2(S^1) \subset L^2(S^1)$

or

$H = \ell_2$

$$\|x\| \overset{\text{standard}}{=} \left(\sum_{-\infty}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$$

$$\|x\|_1 = \left(\sum_{-\infty}^{\infty} |\lambda_i|^2 |x_i|^2 \right)^{\frac{1}{2}}$$

λ_i increasing unbounded both "ways" and "mild" $\left| \frac{\lambda_{2N}}{\lambda_N} \right| \leq C$

③ $B(H)$ = bounded operators

$$B(H:H_1) = \{X \in B(H) \mid X|_{H_1} \in B(H_1),\ X^*|_{H_1} \in B(H_1)\}$$

adjoint in H.

$$U(H:H_1) = B(H:H_1) \cap U(H)$$

Set $S(H:H_1) = \{L : H_1 \longrightarrow H \mid$ Fredholm, of index 0, $\langle \varphi, L\psi\rangle = \langle L\varphi, \psi\rangle\}$

Ensures S has compl. orth-normal system of e-vectors

$H = H^+ \oplus H^-$ from the λ_i (*)

$H_1 = H_1^+ \oplus H_1^-$

L gives $H = H^+(L) \oplus H^-(L)$

$H_1 = H_1^+(L) \oplus H_1^-(L)$

$$S_*(H:H_1) = \{L \in S \mid \exists u \in U(H:H_1) \text{ with } (H^\pm(L), H_1^\pm(L)) \xrightarrow{u} (H^\pm, H_1^\pm)\}$$

Proposition: (Kuipers)

$U(H:H_1)$ is contractible

④ Proposition (A–S)

$\Omega_*(H : H_1) \simeq U(\infty)$

A family of operators of Ω-type is a pair of Hilbert bundles

$$\begin{array}{c} H \supset H_1 \\ \downarrow \\ Q \end{array}$$

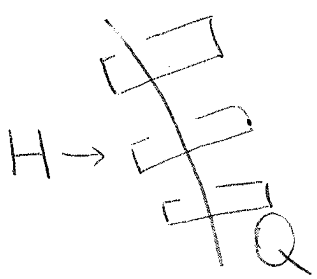
associated to a $U(H:H_1)$ bundle, plus a family of operators $L : H_1 \longrightarrow H$

a section of the associated bundle with fiber $\Omega_*(H : H_1)$

a family of Ω-type

We can define $\overline{H}_*(Q, L)$

– depends only on homotopy, $Q \longrightarrow U(\infty)$
– definition uses Morse theory of Floer type



Suppose $H : H_1$ is a product. Coose

1) f – Morse function on Q
2) given L

Arrange, for $a \in Crit(f)$
L_a has simple spectrum

$\overline{C}_*(Q, L) \rightsquigarrow$ generators indexed by pairs $\alpha = (a, \lambda)$

(5) $a \in Crit(f) \in Q$

$\lambda \in Spec(L_a)$

$M(\alpha, \beta)$; $\alpha = (a, \lambda)$

$\beta = (b, \mu)$

is pairs $(\gamma, [\varphi])$

where $\gamma \in M(a, b)$ on Q

$\gamma : \mathbb{R} \longrightarrow Q$

$\varphi : \mathbb{R} \longrightarrow H$ has $\dot{\varphi} + L_{\gamma(t)} \varphi = 0$

$\varphi \sim e^{-\lambda t} v_\lambda$ as $t \longrightarrow -\infty$

$\varphi \sim e^{-\mu t} v_\mu$ as $t \longrightarrow +\infty$

$[\varphi] := \{ c\varphi \mid c \in \mathbb{C}^* \}$

$c_1(S)$ torsion on Y

Flat connections A in $\det(S)$ is

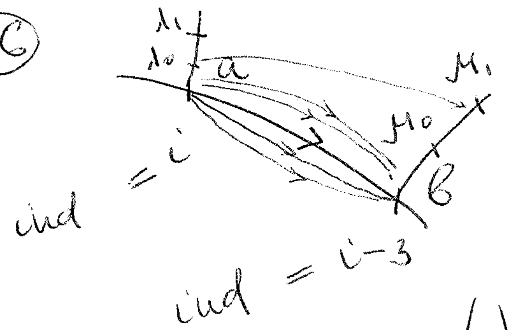
param. by $T^{b_1(Y)}$ $(= "Q")$

With a family of Dirac operators on Y

$\overline{HM}(Y, s) = \overline{H}_*(T, D)$

E.g. $L = L_0$ trivial family

$H_*(Q, L) = H_*(Q) \otimes \mathbb{Z}[T^{-1}, T]$



$M(\alpha, \beta)$?

$\alpha = (a, \lambda_0)$, $\beta = (b, \mu_1)$

(λ_0, μ_0 1st pos. eigenvalues)

Which $\mathbb{T}^{b_1(Y)} \longrightarrow U(\infty)$
classifies the Dirac family ?

Calculate : $ch(\mathrm{ind}(D)) \in H^*(\mathbb{T}, \mathbb{Q})$

$ch : K^{-1}(\mathbb{T}) \hookrightarrow H^{odd}(\mathbb{T}, \mathbb{Q})$

Answer: Factors through

$z : \mathbb{T} \longrightarrow S^3 = SU(2) \hookrightarrow U(\infty)$

z^* (gen of S^3) in H^3

Let a_i be a basis of $H^1(Y)$
Let a_i^* be a dual basis of $H^1(\mathbb{T})$

$z^*(\text{gen}) = \xi \in H^3(\mathbb{T})$ is

$\sum \xi_{ijk}\, a_i^* \cup a_j^* \cup a_k^*$

$\xi_{ijk} = \langle a_i \cup a_j \cup a_k, [Y] \rangle$

cor: Cup-product on Y determines
$\overline{HM}_*(Y, s)$

⑦ Answer: Rank of $\overline{HM}_*(Y, \xi)$ as $\mathbb{R}[T, T^{-1}]$-module is $\dim_{\mathbb{R}} H^*(\Omega^*(T), d+\xi_{DR})$

ξ_{DR} a de Rham rep of ξ