Vertex Operators

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1 The Big Picture

$$\begin{array}{c}\{\text{String theory}\}\\ \uparrow\downarrow \\ \{\text{2-}d\text{ conformal field theory}\}\\ \| \\ \{\text{Vertex operator algebras}\}\\ \uparrow\downarrow \\ \{\text{Modular forms}\}\\ \uparrow\downarrow \\ \{\text{L-series and zeta-functions}\}\end{array}$$

2 Introduction

2.1 Notation

$\mathbb{C}$ = *complex numbers.*
Linear spaces V are defined over $\mathbb{C}$; linear transformations are $\mathbb{C}$-linear; End(V) is the space of *all* endomorphisms of V.
For an indeterminate z,

$$\begin{aligned} V[[z,z^{-1}]] &= \{\sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V\} \\ V[[z]][z^{-1}] &= \{\sum_{n=-M}^{\infty} v_v z^n\} \quad (\text{‘Laurent series’}) \end{aligned}$$

For integers m, n with $n \geq 0$ and m *arbitrary*, define

$$\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{n!}$$

2.2 Local fields

We deal with formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]]. \tag{1}$$

For $v \in V$ we write

$$a(z)v = \sum_{n \in \mathbb{Z}} a_n(v) z^{-n-1} \in V[[z, z^{-1}]].$$

Thus defined, $a(z)$ may be construed as a linear map

$$a(z) : V \to V[[z, z^{-1}]].$$

The endomorphisms a_n are called the *modes* of $a(z)$.

Remark 2.1 *The convention for powers of z in (1) is standard among mathematicians. A different convention is common in the physics literature. Whenever a mathematician and physicist meet to discuss fields, they should first agree on their conventions.*

Definition 2.2 *$a(z) \in End(V)[[z, z^{-1}]]$ is a* field *if it satisfies the following truncation condition $\forall v \in V$:*

$$a(z)v \in V[[z]][z^{-1}].$$

I.e. for each $v \in V$ there is an integer N (depending on v) such that $a_n(v) = 0$ for all $n > N$.

Set

$$\mathfrak{F}(V) = \{a(z) \in \mathrm{End}(V)[[z, z^{-1}]] \mid a(z) \text{ is a field}\}.$$

$\mathfrak{F}(V)$ is the field-theoretic analog of $\mathrm{End}(V)$. It is a subspace of $\mathrm{End}(V)[[z, z^{-1}]]$.

The introduction of a *second* indeterminate facilitates the study of products and commutators of fields. Set

$$[\sum a_m z_1^{-m-1}, \sum b_n z_2^{-n-1}] = \sum [a_m, b_n] z_1^{-m-1} z_2^{-n-1},$$

which is an element of $\mathrm{End}(V)[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$.

The idea of *locality* is crucial.

Definition 2.3 $a(z), b(z) \in End(V)[[z, z^{-1}]]$ *are called* mutually local *if there is a nonnegative integer k such that*

$$(z_1 - z_2)^k [a(z_1), b(z_2)] = 0. \tag{2}$$

If (2) holds, write $a(z) \sim_k b(z)$ and say that $a(z)$ and $b(z)$ are *mutually local of order k*. Write $a(z) \sim b(z)$ if k is not specified. $a(z)$ is *local* if $a(z) \sim a(z)$. (2) means that the coefficient of each monomial $z_1^{r-1} z_2^{s-1}$ in the expansion of the lhs vanishes, that is

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} [a_{k-j-r}, b_{j-s}] = 0. \tag{3}$$

Locality is a *symmetric relation*, but in general is neither reflexive nor transitive.

Exercise: Let $\partial a(z) = \sum (-n-1) a_n z^{-n-2}$ be the formal derivative of $a(z)$. Suppose that $a(z), b(z) \in \mathfrak{F}(V)$ and $a(z) \sim_k b(z)$. Prove that $\partial a(z) \in \mathfrak{F}(V)$ and $\partial a(z) \sim_{k+1} b(z)$.

2.3 Axioms for a vertex algebra

Definition 2.4 *: A* vertex algebra *is a quadruple* $(V, Y, \mathbf{1}, D)$ *where*

$$Y : V \to \mathfrak{F}(V),\ v \mapsto Y(v, z) = \sum v_n z^{-n-1} \text{ is a linear map}$$
$$\mathbf{1} \in V,\ \mathbf{1} \neq 0$$
$$D : V \to V \text{ and } D\mathbf{1} = 0.$$

The following axioms hold $\forall u, v \in V$*:*

$$\textit{locality} : Y(u, z) \sim Y(v, z)$$
$$\textit{creativity} : Y(u, z)\mathbf{1} = u + O(z)$$
$$\textit{translation-covariance} : [D, Y(u, z)] = \partial Y(u, z)$$

Remark 2.5 *(a) Elements of V are the* states *and V the* state space, or Fock space*;* $\mathbf{1}$ *the* vacuum*.*
(b) Y is the state-field correspondence*. It is a linear injection.*

(c) Creativity says that $Y(u,z)\mathbf{1} \in V[[z]]$ and that $Y(u,z)$ creates *the state u from the vacuum. In terms of modes,*

$$\begin{aligned} u_n\mathbf{1} &= 0, \ n \geq 0 \\ u_{-1}\mathbf{1} &= u \end{aligned}$$

(d) The modal version of translation-covariance is

$$[D, u_n] = -nu_{n-1}$$

(e)This set-up models the creation and annihilation of (bosonic) states from the vacuum. The axioms are elegant but opaque. All of the subtlety is tied to locality and its consequences.

Exercise: Prove that

$$Y(u,z)\mathbf{1} = e^{zD}u \ (= \sum_{n\geq 0} \frac{z^n D^n u}{n!})$$

Theorem 2.6 *Let $(V, \mathbf{1}, D)$ be as above. Suppose $S \subseteq \mathfrak{F}(V)$ is a set of mutually local, creative, translation-covariant fields which* generates *V in the sense that*

$$V = span\{a^1_{-n_1} \dots a^k_{-n_k}\mathbf{1} \mid a^i(z) \in S, n_1, \dots, n_k \geq 1, k \geq 0\}.$$

There is a unique vertex algebra $(V, Y, \mathbf{1}, D)$ such that $Y(a^i_{-1}\mathbf{1}, z) = a^i(z)$.

To construct vertex algebras, we thus need only look for generating sets of mutually local, creative, translation-covariant fields. We discuss two fundamental examples where S consists of a *single* field.

2.4 Heisenberg algebra

$A = \mathbb{C}a$ is a 1-dimensional linear space. The *affinization* of A is the Lie algebra

$$\hat{A} := A[t, t^{-1}] \oplus \mathbb{C}K$$

with brackets

$$\begin{aligned} [a \otimes t^m, a \otimes t^n] &= m\delta_{m,-n}K \\ [\hat{A}, K] &= 0. \end{aligned}$$

Remark 2.7 *Set* $p_m = (1/\sqrt{m})a \otimes t^m (m > 0)$ *and* $q_{-m} = (1/\sqrt{-m})a \otimes t^m (m < 0)$. *Then*

$$[p_m, q_n] = \delta_{m,n} K, \tag{4}$$

which is essentially the canonical commutator relations *of QM.*

$\hat{A}^{\geq} := \langle a \otimes t^n, K \mid n \geq 0 \rangle \subseteq \hat{A}$ is a Lie subalgebra. Let $\mathbb{C}v_h$ be a 1-dimensional $\hat{A}^{\geq}$-module via

$$\begin{aligned} K.v_h &= v_h \\ (a \otimes t^0).v_h &= hv_h \\ (a \otimes t^n).v_h &= 0 \quad (n \geq 1). \end{aligned}$$

Here, $h \in \mathbb{C}$ is arbitrary. Let $\mathcal{U}(L)$ denote the *universal enveloping algebra* of a Lie algebra L. The induced (Verma) module is

$$M_h := \mathrm{Ind}_{\mathcal{U}(\hat{A}^{\geq})}^{\mathcal{U}(\hat{A})} \mathbb{C}v_h = \mathcal{U}(\hat{A}) \otimes_{\mathcal{U}(\hat{A}^{\geq})} \mathbb{C}v_h.$$

It is an $\hat{A}$-module (or $\mathcal{U}(\hat{A})$-module) with basis

$$\{a_{-n_1} \dots a_{-n_k} \otimes v_h \mid n_1 \geq \cdots \geq n_k \geq 1\}, \; (a_{-n} = a \otimes t^n).$$

(See the Appendix for further background.)

Exercise: Let $a_n \in \mathrm{End}(M_h)$ be the induced action of $a \otimes t^n$ on M_h, with $a(z) = \sum a_n z^{-n-1}$. Prove that

$$\begin{aligned} a(z) &\in \mathfrak{F}(M_h) \\ a(z) &\sim_2 a(z) \end{aligned}$$

Solution: Fix $v = a_{-n_1} \dots a_{-n_k} \otimes v_h$ with $n_1 \geq \cdots \geq n_k \geq 1$. If $n > n_1$ then a_n commutes with all modes a_{-n_i} by the bracket relations for $\hat{A}$. Therefore, $a_n.v = a_n.a_{-n_1} \dots a_{-n_k} \otimes v_h = a_{-n_1} \dots a_{-n_k} \otimes a_n.v_h = 0$. This shows that $a(z) \in \mathfrak{F}(M_h)$. As for locality, (cf. (3)) we have

$$\begin{aligned} & \sum_{j=0}^{2} (-1)^j \binom{2}{j} [a_{2-j-r}, a_{j-s}] \\ = & \sum_{j=0}^{2} (-1)^j \binom{2}{j} (2-j-r) \delta_{2-j-r,s-j} K \\ = & \{(2-r) - 2(1-r) - r\} \, \delta_{r+s,2} K = 0. \qquad \square \end{aligned}$$

Using Theorem 2.6 with $A = \{a(z)\}$, one can prove

Theorem 2.8 *$M = M_0$ is a vertex algebra, generated by $a(z)$ and with vacuum vector v_0.*

Remark 2.9 *In M_0 the CCR (4) read $[p_m, q_n] = \delta_{m,n} Id$. They may be realized by taking $p_m = \frac{\partial}{\partial x_{-m}}, q_n = x_{-n}$ acting on $\mathbb{C}[x_{-1}, x_{-2}, \ldots]$ in the usual way. This affords an alternate way to understand the Heisenberg theory.*

2.5 Virasoro algebra

The Virasoro algebra is the Lie algebra with underlying linear space

$$Vir = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \mathbb{C} K$$

and bracket relations

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m,-n} K$$

$Vir^{\geq} := \oplus_{n \geq 0} \mathbb{C} L_n \oplus \mathbb{C} K \subseteq Vir$ is a subalgebra. Let $\mathbb{C} v_{c,h}$ be a 1-dimensional space made into a $Vir^{\geq}$-module via

$$\begin{aligned} K.v_{c,h} &= c v_{c,h} \\ L_0.v_{c,h} &= h v_{c,h} \\ L_n.v_{c,h} &= 0 \quad (n \geq 1). \end{aligned}$$

Here, $c, h \in \mathbb{C}$ are arbitrary. The induced (Verma) module is

$$M_{c,h} = \mathcal{U}(Vir) \otimes_{\mathcal{U}(Vir^{\geq})} \mathbb{C} v_{c,h}.$$

$M_{c,h}$ is a Vir-module with basis

$$\{L_{-n_1} \ldots L_{-n_k} \otimes v_{c,h} \mid n_1 \geq \cdots \geq n_k \geq 1\}.$$

(See Appendix for further background.)

Exercise: Identify elements of Vir with the endomorphisms they induce on $M_{c,h}$. Let $\omega(z) = \sum L_n z^{-n-2}$. Prove that

$$\begin{aligned} &\omega(z) \in \mathfrak{F}(M_{c,h}) \\ &\omega(z) \sim_4 \omega(z) \end{aligned}$$

With a minor modification, we find as before

Theorem 2.10 *Let $Vir_c = M_{c,0} / \mathcal{U}(Vir) L_{-1} v_{c,0}$. Vir_c is a vertex algebra, generated by $\omega(z)$ and with vacuum vactor $v_{c,0}$.*

2.6 Axioms for a Vertex Operator Algebra

A vertex operator algebra is a special type of vertex algebra with a dedicated Virasoro field. Precisely

Definition 2.11 *: A* vertex operator algebra *is a quadruple* $(V, Y, \mathbf{1}, \omega)$ *where* $V = \oplus_{n\in\mathbb{Z}} V_n$ *is a* $\mathbb{Z}$*-graded linear space and*

$$Y : V \to \mathfrak{F}(V),\ v \mapsto Y(v, z) = \sum v_n z^{-n-1}$$
$$\mathbf{1}, \omega \in V,\ \mathbf{1} \neq 0.$$

The following axioms hold $\forall u, v \in V$*:*

$$\textit{locality} : Y(u, z) \sim Y(v, z)$$
$$\textit{creativity} : Y(u, z)\mathbf{1} = u + O(z)$$
$$\dim V_n < \infty,\ V_n = 0 \textit{ for } n \ll 0$$
$$Y(\omega, z) = \sum L_n z^{-n-2} \textit{ with a constant } c \textit{ such that}$$
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n} c Id$$
$$V_n = \{v \in V_n \mid L_0 v = nv\}$$
$$Y(L_{-1}u, z) = \partial Y(u, z)$$

There are a number of more-or-less direct consequences of the axioms. Here are a few.

$$\begin{aligned}
Y(\mathbf{1}, z) &= \mathrm{Id} \\
\mathbf{1} &\in V_0 \\
\omega &\in V_2 \\
L_n \mathbf{1} &= 0 \text{ for } n \geq -1 \\
[L(-1), Y(u, z)] &= \partial Y(u, z) \\
[u_m, v_n] &= \sum_{i\geq 0} \binom{m}{i} (u_i v)_{m+n-i} \text{ (commutator)} \\
(u_m v)_n &= \sum_{i\geq 0} (-1)^i \binom{m}{i} \{u_{m-i} v_{n+i} - (-1)^m v_{m+n-i} u_i\} \text{ (associator)} \\
u_m v &= \sum_{i\geq 0} (-1)^{m+i+1} \frac{1}{i!} L_{-1}^i v_{m+i} u \text{ (skew-symmetry)}
\end{aligned}$$

Remark 2.12 *(a) The constant c is the* central charge *of V. It is an important invariant of V.*
(b) ω is called the Virasoro *or* conformal *vector.*
(c) L_{-1} plays the rôle of D.
(d) All of the sums that occur in the last display are finite *(why?)*
(e) The last three identities are all special cases of the following, which holds $\forall u, v, w \in V, \forall p, q, r \in \mathbb{Z}$:

$$\sum_{i\geq 0} \binom{p}{i} (u_{r+i}v)_{p+q-i}w =$$
$$\sum_{i\geq 0} (-1)^i \binom{r}{i} \{u_{p+r-i}v_{q+i}w - (-1)^r v_{q+r-i}u_{p+i}w\} \qquad (5)$$

This identity together with $Y(\mathbf{1}, z) = Id_V$ may be taken as an equivalent *way to define vertex algebra.*

Exercise: Deduce the commutator, associator, and skew-symmetry formulas from (5).

Examples
1. The Heisenberg theory M_0 is a vertex operator algebra with $\omega = 1/2a_{-1}^2\mathbf{1} = 1/2a_{-1}a = 1/2a_{-2}\mathbf{1}$. This is the physicist's 1-*d bosonic string*.

2. The Virasoro algebra Vir_c is a vertex operator algebra with $\omega = L_{-2}\mathbf{1}$.

3 Modular forms

Elliptic modular forms (rather than zeta functions) provide the initial connection between vertex operator algebras and arithmetic.

3.1 Modular forms

Notation:

$$\begin{aligned}
\Gamma &= SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\} \\
&= \left\langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \\
\mathfrak{H} &= \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \\
q &= e^{2\pi i \tau} \\
\Gamma \times \mathfrak{H} &\to \mathfrak{H},\ \gamma.\tau = \frac{a\tau+b}{c\tau+d} \\
f|_k\gamma(\tau) &= (c\tau+d)^{-k} f(\gamma\tau)\ (k \in \mathbb{Z},\ f \text{ meromorphic in } \mathfrak{H}).
\end{aligned}$$

A *weak modular form* of weight k on Γ is a meromorphic function $f(\tau)$ such that $f|_k\gamma(\tau) = f(\tau)$ for all $\gamma \in \Gamma$. This amounts to

$$\begin{aligned}
f(T\tau) &= f(\tau+1) = f(\tau) \\
f(S\tau) &= f(-1/\tau) = \tau^k f(\tau)
\end{aligned}$$

The first of these equations implies that $f(\tau)$ has a *q-expansion*

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n.$$

Remark 3.1 *The interpretation of the* Fourier coefficients a_n *of modular forms in terms of data originating from vertex operator algebras lies at the heart of the applications of modular forms to CFT.*

$f(\tau)$ is a *meromorphic* (resp. *holomorphic*) modular form of weight k if its q-expansion has the form

$$f(\tau) = \sum_{n\geq n_0} a_n q^n$$

for some n_0 (resp. for $n_0 \geq 0$), and if (in the holomorphic case) $f(\tau)$ is also holomorphic in $\mathfrak{H}$. We say in this case that $f(\tau)$ has a singularity (*pole* or *zero*) of order n_0 at ∞.

More generally we consider subgroups $\Gamma_1 \subseteq \Gamma$ of *finite* index. Then $f(\tau)$ is a meromorphic (holomorphic) modular form of weight k on Γ_1 if $f|_k\gamma(\tau) = f(\tau)\ \forall \gamma \in \Gamma_1$ and there are q-expansions

$$f|_k\gamma(\tau) = \sum_{n \geq n_0} a_n q^{n/N} \quad \forall \gamma \in \Gamma.$$

as before. Here, $N \geq 1$ such that $T^N \in \Gamma_1$ is the *level* of f.

3.2 j-function, eta-function, theta series

(a) The set of all modular forms of weight 0 form a field which is a simple transcendental extension of $\mathbb{C}$. Indeed

$$\{\text{modular forms of weight } 0\} = \mathbb{C}(j)$$

where j satisfies

$$j(\tau) = q^{-1} + 744 + 196884q + \ldots$$

and is holomorphic in $\mathfrak{H}$. This is the famous *j-function.*

The set of holomorphic modular forms of weight k on Γ_1 is a finite-dimensional linear space. It is 0 for $k < 0$ and consists of the constants $\mathbb{C}$ if $k = 0$. We give just a few examples.

(b) The eta-function may be defined by its q-expansion

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

It is not technically a modular form as we have defined it (formally, it has weight 1/2), but an even power $\eta(\tau)^{2k}$ has weight k, is holomorphic throughout $\mathfrak{H}$, and is a holomorphic modular form if, and only if, $k \geq 0$. Note that

$$\eta(\tau)^{-1} = q^{-1/24} \sum_{n \geq 0} p(n) q^n = q^{-1/24}(1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots)$$

where $p(n)$ is the *unrestricted partition function.*

(c) An *even* lattice is a f.g. free abelian group $L \cong \mathbb{Z}^l$ equipped with a positive-definite integral bilinear form

$$(\ , \ ) : L \times L \to \mathbb{Z}$$

such the associated quadratic form $(\alpha, \ \alpha)/2$ is also integral. The *theta-function* of L is the formal q-expansion

$$\theta_L(\tau) = \sum_{\alpha \in L} q^{(\alpha,\alpha)/2}.$$

It turns out (Hecke-Schoeneberg) that l is even and that θ_L is a holomorphic modular form of weight $l/2$ on a certain subgroup of Γ. θ_L is a modular form of level 1 (i.e. on the full modular group Γ) if, and only if, L is *self-dual*. This means that the Gram matrix for a $\mathbb{Z}$-basis of L is *unimodular* (determinant ± 1). In this case l is necessarily divisible by 8. The (unique) example with $l = 8$ is the E_8 *root lattice*, where

$$\begin{aligned} \theta_{E_8}(\tau) &= 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + \ldots \\ \sigma_3(n) &= \sum_{d|n} d^3. \end{aligned}$$

There is a web of relations among the modular forms which mathematicians (and, more recently, also physicists) have been studying for 150 years. A relatively easy one is

$$j(\tau) = \frac{\theta_{E_8}(\tau)^3}{\eta(\tau)^{24}} = q^{-1} \left(1 + 240 \sum \sigma_3(n) q^n\right)^3 \left(\sum p(n) q^n\right)^{24}$$

This gives an explicit closed formula for the Fourier coefficients of j. It is hard to use, but at least shows that they are nonnegative integers (and growing exponentially).

3.3 Mellin transform

For a holomorphic modular form $f(\tau) = \sum_{n \geq 1} a_n q^n$ of weight k, the *Mellin transform* of f is a certain integral transform that produces an L-function

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Indeed,

$$\int_0^\infty f(it)t^{s-1}dt = (2\pi)^{-s}\Gamma(s)L(f,s).$$

There is an estimate $a_n = O(n^k)$ for $n \to \infty$ that yields the absolute convergence of $L(f,s)$ in the right half-plane Re $s > 1+k$. This, in brief, is how modular forms relate to zeta functions.

4 Partition functions

Here we begin to relate vertex operator algebras to modular forms.

4.1 Zero modes and the partition function of a vertex operator algebra

Let

$$V = \oplus V_n$$

be a vertex operator algebra of central charge c. A basic consequence of the axioms is the following fact: if $v \in V_k$ (we write $wt(v) = k$) then

$$v_m : V_n \to V_{n+wt(v)-m-1}. \tag{6}$$

The *zero* mode of v is the mode

$$o(v) := v_{wt(v)-1}.$$

It is an operator of weight zero in the sense that

$$o(v) : V_n \to V_n.$$

One can thus form the *formal* q-expansion, or q-trace

$$Z_V(v,q) := \mathrm{Tr}_V q^{L(0)-c/24} o(v) = q^{-c/24} \sum \mathrm{Tr}_{V_n} o(v) q^n$$

(where we have used $L(0)v = nv$ for $v \in V_n$).

We consider here the case when $v = \mathbf{1}$. The general case will be considered later. Since $Y(\mathbf{1}, z) = \mathrm{Id}$ and $\mathbf{1} \in V_0$ then $o(\mathbf{1}) = \mathrm{Id}$. Hence

$$Z_V(q) := Z_V(\mathbf{1}, q) = q^{-c/24} \sum \dim V_n q^n.$$

This is the *partition function* of V.

Example 1: Heisenberg theories The underlying linear space for the Heisenberg vertex operator algebra $M = M_0$ (cf. Theorem 2.8) has basis $\{v = a_{-n_1} \ldots a_{-n_k}\mathbf{1}\}$ where $n_1 \geq \ldots \geq n_k \geq 1$ ranges over increasing sequences of positive integers, i.e. (unrestricted) partitions. Moreover $wt(v) = n_1 + \ldots + n_k$, so that $\dim V_n = p(n)$. Since $c = 1$ in this case then

$$Z_M(q) = q^{-1/24} \sum p(n) q^n = \eta(\tau)^{-1}.$$

Given vertex operator algebras V, W, the tensor product $V \otimes W$ acquires the structure of a vertex operator algebra with the natural tensor product grading and with central charge the *sum* of the central charges of V and W. The tensor product $M^{\otimes n}$ is called the *rank* n Heisenberg vertex operator algebra, or n-dimensional bosonic string. Thus

$$Z_{M^{\otimes n}}(q) = Z_M(q)^n = \eta(\tau)^{-n}$$

is a modular form of weight $-n/2$.

Example 2: Virasoro theories (cf.Theorem 2.9). The space $M_{c,0}$ used to construct the Virasoro vertex operator algebra Vir_c has a basis consisting of $\{v = L_{-n_1} \ldots L_{-n_k} \otimes v_{c,0} \mid n_1 \geq \ldots \geq n_k \geq 1\}$; moreover $wt(v) = n_1 + \ldots n_k$. Thus the $\mathbb{Z}$-grading on this space is $\sum p(n) q^n = \prod (1 - q^n)^{-1}$. It follows that

$$Z_{Vir_c}(q) = \frac{q^{(1-c)/24}(1-q)}{\eta(q)} = q^{-c/24} \prod_{n=2}^{\infty} (1 - q^n)^{-1}.$$

This is *not* a modular form.

4.2 Lattice theories

Let L be an even lattice of rank l (cf. Subsection 3.2). The *lattice vertex operator algebra* V_L has underlying linear space

$$M^{\otimes l} \otimes \mathbb{C}[L] = \oplus_{\alpha \in L} M^{\otimes l} \otimes e^{\alpha} \qquad (7)$$

and central charge l. Here, $\mathbb{C}[L] = \oplus_{\alpha \in L} \mathbb{C} e^{\alpha}$ is the *group algebra* of L, with (integral) grading such that e^{α} has weight $(\alpha, \alpha)/2$. $M^{\otimes l} = M^{\otimes l} \otimes e^0 \subseteq V_L$ is a *subvertex operator algebra*, and in particular the Virasoro vector of $M^{\otimes l}$ is also the Virasoro vector for V_L. We will not give the details of the construction of V_L here. The main task is to define vertex operators $Y(\mathbf{1} \otimes e^{\alpha}, z)$ for $\alpha \in L$ so that they are mutually local with each other and with the fields $Y(u, z), u \in M^{\otimes l}$. Then Theorem 2.6 can be applied.

The partition function is readily calculated as follows:

$$Z_{V_L}(q) = Z_{M^{\otimes l}} \sum_{\alpha \in L} q^{\langle \alpha, \alpha \rangle / 2} = \frac{\theta_L(\tau)}{\eta(\tau)^l}.$$

Both $\theta_L(\tau)$ and $\eta(\tau)^l$ are (holomorphic) modular forms of weight $l/2$ for some finite-index subgroup of Γ, and their quotient is a modular function of weight 0.

5 Modular-invariance I

The examples in the last Subsection illustrate the fact that the partition function $Z_V(q)$ of a vertex operator algebra V may, or may not, have modular properties. An important class of vertex operator algebras, called *rational vertex operator algebras*, are expected to have strong modular-invariance properties. We describe this situation and some of what is known about it in this Section.

5.1 Modules over a vertex operator algebra

The relationship between a vertex operator algebra V and a V-module is analogous to that of a Lie algebra L and an L-module. It is useful to distinguish various types of V-module.

Definition 5.1 *A* weak *V-module is a pair (M, Y_M) consisting of a linear space M and a linear map $Y_M : V \to \mathfrak{F}(M)$, $v \mapsto Y_M(v, z) = \sum v_n^M z^{-n-1}$ such that $Y_M(\mathbf{1}, z) = Id_M$ and $\forall u, v \in V$ we have*

$$\sum_{i \geq 0} \binom{p}{i} (u_{r+i} v)^M_{p+q-i} = \sum_{i \geq 0} (-1)^i \binom{r}{i} \left\{ u^M_{p+r-i} v^M_{q+i} - (-1)^r v^M_{q+r-i} u^M_{p+i} \right\}.$$

This is the natural analog of (5). A weak V-module is essentially a module for a vertex algebra.

Definition 5.2 *An* admissible V*-module is a weak* V*-module* (M, Y_M) *equipped with an* $\mathbb{N}$*-grading* $M = \oplus_{n\geq 0} M_n$ *such that*

$$v_n^M : M_p \longrightarrow M_{p+wt(v)-n-1}.$$

This is the analog of (6).

Definition 5.3 *A* V*-module is a weak* V*-module* (M, Y_M) *equipped with a grading* $M = \oplus_{\lambda\in\mathbb{C}} M_\lambda$ *such that*

$$\begin{aligned}
&\dim M_\lambda < \infty \\
&\forall\lambda, M_{\lambda+n} = 0 \textit{ for } n \ll 0 \\
&L_0 m = \lambda m,\ m \in M_\lambda
\end{aligned}$$

We have containments

$$\{\text{weak } V\text{-modules}\} \supseteq \{\text{admissible } V\text{-modules}\} \supseteq \{V\text{-modules}\},$$

which implicitly means that V-modules can be equipped with an $\mathbb{N}$-grading making them admissible. Note that V-modules are not necessarily $\mathbb{Z}$-graded. We can define the *partition function* of a V-module as before, that is

$$Z_M(q) := Tr_M q^{L_0 - c/24} = q^{-c/24} \sum_\lambda \dim M_\lambda q^\lambda.$$

A V-module is called *irreducible* if no proper, nonzero subspace of V is invariant under all modes v_n. For an irreducible module, the L_0-grading always takes the form $M = \oplus_{n\geq 0} M_{h+n}$ for some scalar h, called the *conformal weight* of M. Thus in this case

$$Z_M(q) = q^{h-c/24} \sum_{n\geq 0} \dim M_{h+n} q^n.$$

V is called *simple* if the *adjoint* module V is itself an irreducible V-module.

Examples:

1. For every $h \in \mathbb{C}$, the Verma module M_h (cf. Subsection 2.4) is an irreducible module for the Heisenberg vertex operator algebra M_0 of conformal weight h. In particular, M_0 is simple. We have

$$Z_{M_h}(q) = q^{h-1/24} \sum p(n) q^n = \frac{q^h}{\eta(q)}.$$

2. For all $c, h \in \mathbb{C}$, the Verma module $M_{c,h}$ for the Virasoro algebra (cf. Subsection 2.5) is a module (not necessarily irreducible) for Vir_c. It has the same partition function as the previous example.

5.2 Rational vertex operator algebras

Definition 5.4 *A vertex operator algebra V is called* rational *if every admissible V-module is completely reducible, i.e. a direct sum of irreducible V-modules.*

Theorem 5.5 *Suppose that V is a rational vertex operator algebra. Then V has only finitely many (inequivalent) irreducible V-modules, and every admissible, irreducible module is an (ordinary) V-module.*

Example 5.6 *The Heisenberg theory M_0 has infinitely many irreducibe modules (cf. Example 1 in the previous Subsection). Thus M_0 is* not *rational.*

Definition 5.7 *Let $C_2(V) = \langle u_{-2}v \mid u, v \in V \rangle$. We say that V is* C_2-cofinite *if $C_2(V)$ has finite codimension in V.*

Theorem 5.8 *Suppose that V is a vertex operator algebra that is C_2-cofinite. Then V has only finitely many (inequivalent) irreducible V-modules.*

Remark 5.9 *It is widely believed that every rational vertex operator algebra is C_2-cofinite. This is presently an important open question. The converse is false: these are the so-called* logarithmic field theories*. They are vertex operator algebras with only finitely many irreducible modules, but for which complete reducibility of modules fails. Thus far, such theories have been mainly (though not exclusively) investigated by physicists.*

The following omnibus theorem collects some of the main facts about vertex operator algebras which are *regular*, that is they are *both* rational and C_2-cofinite.

Theorem 5.10 *Suppose that V is a regular vertex operator algebra of central charge c. Let $M^1, \ldots, M^r$ be the irreducible V-modules, and let h^i be the conformal weight of M^i. Then the following hold:*
(a) c and each h^i are rational *numbers*
(b) The partition functions $Z_{M^i}(q)$ are holomorphic in $\tau \in \mathfrak{H}$ (cf. Subsection

3.1).
(c) The space E spanned by the $Z_{M^i}(\tau)$ is modular-invariant *in the sense that for $\gamma \in \Gamma$ there are scalars γ_{ij} so that*

$$Z_{M^i}(\gamma\tau) = \sum \gamma_{ij} Z_{M^j}(\tau).$$

In other words, $\gamma \mapsto (\gamma_{ij})$ is a representation *of Γ on E.*

Remark 5.11 *One expects that in this situation,* each *$Z_{M^i}(\tau)$ is a modular function of weight 0 on a subgroup of Γ of finite index. There are 'physical' proofs of this in the literature, but a complete mathematical proof is yet to be found.*

Examples:
1. Lattice theories: lattice theories V_L (cf. Subsection 4.2) are regular vertex operator algebras. Let L have rank l and let $L^0 := \{\alpha \in \mathbb{R}^l \mid (\alpha, \beta) \in \mathbb{Z}\ \forall \beta \in L\}$ be the *dual lattice.* Then $L \subseteq L^0$ has *finite index*, and the cosets $\alpha + L,\ \alpha \in L^0$, index the irreducible V-modules. Indeed, the underlying linear spaces for the irreducible modules are

$$V_{L+\alpha} := M^{\otimes l} \otimes \mathbb{C}[L + \alpha] = \oplus_{\beta \in L} M^{\otimes l} \otimes e^{\beta+\alpha}.$$

(Compare with (7).) The conformal weight of this module is $1/2(\alpha, \alpha)$, and the partition function is

$$Z_{V_{L+\alpha}}(q) = \frac{\theta_{L+\alpha}(\tau)}{\eta(\tau)^l} = \eta(q)^{-l} \sum_{\beta \in L} q^{(\beta+\alpha,\beta+\alpha)/2},$$

which is indeed a modular form of weight 0, holomorphic in $\mathfrak{H}$.
2. Virasoro discrete series: The Virasoro vertex operator algebra Vir_c is not necessarily simple, but there is a *unique* maximal proper submodule whose quotient is a simple vertex operator algebra L_c of central charge c. It turns out that L_c is rational if, and only if, c lies in the so-called *discrete series.* This means that there are coprime integers $p, q \geq 2$ such that

$$c = c_{pq} = 1 - \frac{6(p-q)^2}{pq}.$$

The case $c = c_{3,4} = 1/2$, for example, corresponds to the famous Ising model. This theory and other members of the discrete series (esp. those in the *unitary* discrete series, when $q = p + 1$), describe discrete statistical models at critical exponents. (Cf. ref. 1 for further information.)

5.3 Holomorphic theories

Call a regular vertex operator algebra V *holomorphic* if it has a *unique* irreducible module, namely the adjoint module V. In this case Theorem 5.10 can be refined to give

Theorem 5.12 *Suppose that V is a regular, holomorphic vertex operator algebra of central charge c. Then the following hold:*
(a) c is an integer divisible by 8
(b) The partition function $Z_V(q)$ satisfies

$$Z_V(\gamma\tau) = \chi(\gamma) Z_M(\tau)$$

where $\chi : \Gamma \to \mathbb{C}^*$ is a character of order dividing 3. In particular, $Z_V(q)$ is a modular function on a (normal) subgroup of Γ of index at most 3.

Theories of particular interest in both mathematics and physics are of so-called *CFT-type*, where the vacuum is *nondegenerate*. This means that the grading on V takes the shape

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus \ldots$$

In such cases, if V is also regular and holomorphic (and not equal to $\mathbb{C}\mathbf{1}$) then c is a positive integer multiple of 8. Examples of such theories include (but are not limited to) lattice theories V_L with L *self-dual* (cf. §3.2 and the Example in §5.2). It is known that up to isomorphism, the only regular, holomorphic vertex operator algebra of CFT-type and with $c = 8$ is the E_8 lattice theory, ie L is the root lattice of type E_8. (This is the *only* indecomposable root lattice which is self-dual.) This result is related to the uniqueness of the heterotic string.

As a final example, we mention some recent work of E. Witten. In this work, certain holomorphic vertex operator algebras $V^{(k)}, k = 1, 2, \ldots$ are posited to exist which are related via the AdS-CFT correspondence (here undefined) to phenomena concerning gravity with a negative cosmological constant. $V^{(k)}$ has central charge $c = 24k$ and a *minimal* structure compatible with the requirements of modular-invariance imposed by Theorems 5.10 and 5.12. To be more precise, minimality is the assumption that the first $k+1$ homogeneous pieces $V_0^{(k)}, \ldots, V_k^{(k)}$ of $V^{(k)}$ coincide with those of the Virasoro subtheory $Vir_{24k} \subseteq V^{(k)}$. In other words, the initial segment of $Z_{V^{(k)}}(q)$ consisting of the nonpositive powers of q coincides with that for $Z_{Vir_{24k}}(q)$.

The general existence of such theories is utterly opaque at this time. The lone exception is the famous *Moonshine module* $V^\natural$, which has $c = 24$ corresponding to the case $k = 1$. The partition function is (cf. Subsection 3.2)

$$Z_{V^\natural}(q) = j(q) - 744 = q^{-1} + 0 + 196884q + \ldots$$

Here, the vanishing of the constant term is tantamount to a minimal structure, because $Z_{Vir_{24}}(q) = q^{-1}(1 + 0q + \ldots)$ (cf. Example 2, §4.1).

<u>Exercise</u>: Show that there is *exactly one* modular function $f(q)$ of weight zero on Γ whose first $k+1$ Fourier coefficients coincide with those of $Z_{Vir_{24k}}(q)$ and which satisfies the conditions imposed on $Z_{V^{(k)}}(q)$ by Theorems 5.10 and 5.12. Show that $f(q) = p(j(q))$ where $p(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree k. (Hint: use (a) in §3.2.)

6 Appendix: Lie algebras and representations

An *associative algebra* is a linear space A equipped with a bilinear, associative product $A \otimes A \to A$, denoted by juxtaposition. Thus $a \otimes b \mapsto ab$ and

$$(ab)c = a(bc).$$

A *Lie algebra* is a linear space L equipped with a bilinear product (usually called bracket) $[\ \ ] : L \otimes L \to L$ such that

$$\begin{aligned} &[ab] = -[ba] && \text{(skew-commutativity)} \\ &[a[bc]] + [b[ca]] + [c[ab]] = 0 && \text{(Jacobi identity)} \end{aligned}$$

An associative algebra A gives rise to a Lie algebra A^- on the *same* linear space by defining $[ab] = ab - ba$. A basic example is $\mathrm{End}(V)$ for a linear space V, where the associative product is composition of endomorphisms. This situation can be exploited using another basic associative algebra, the *tensor algebra*

$$T(V) = \oplus_{n \geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus V \otimes V \oplus \ldots$$

over V. Let $\iota : V \to T(V)$ be canonical identification of V with the degree 1 piece of $T(V)$. The *universal mapping property* (UMP) for tensor algebras says that any linear map $f : V \to A$ into an associative algebra A has a *unique* extension to a morphism of associative algebras $\alpha : T(V) \to A$:

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ & \iota \searrow & \uparrow \alpha \\ & & T(V) \end{array}$$

with $f = \alpha \circ \iota$.

A *representation* of a Lie algebra L is a linear map $\pi : L \to \mathrm{End}(V)$ for some V such that

$$\pi([ab]) = \pi(a)\pi(b) - \pi(b)\pi(a).$$

That is, $\pi : L \to \mathrm{End}(V)^-$ is a morphism of Lie algebras. We call V an L-module.

UMP provides an extension of π to a morphism of associative algebras $\alpha : T(L) \to \mathrm{End}(V)$. Identifying $a \in L$ with its image in $T(L)$, we see that for $a, b \in L$

$$\begin{aligned} \alpha(a \otimes b - b \otimes a - [ab]) &= \alpha(a)\alpha(b) - \alpha(b)\alpha(a) - \alpha([ab]) \\ &= \pi(a)\pi(b) - \pi(b)\pi(a) - \pi([ab]) \\ &= 0. \end{aligned}$$

Introduce the 2-sided ideal $J \subseteq T(L)$ generated by $a \otimes b - b \otimes a - [ab],\ a, b \in L$, and set

$$\mathcal{U}(L) = T(L)/J.$$

This is the *universal enveloping algebra* of L. Thus every representation π of L extends to a representation of the universal enveloping algebra in a canonical way:

$$\begin{array}{ccc} L & \xrightarrow{\pi} & \mathrm{End}(V) \\ & \iota' \searrow & \uparrow \alpha \\ & & \mathcal{U}(L) \end{array}$$

where ι' is the composition $L \xrightarrow{\iota} T(L) \to \mathcal{U}(L)$

Theorem 6.1 *(PBW Theorem). Fix an ordered basis $x_1, x_2, \ldots$ of L, with $\bar{x}_i$ the image of x_i in $\mathcal{U}(L)$. Then*

$$\{\bar{x}_{i_1}\bar{x}_{i_2}\ldots\bar{x}_{i_k} \mid i_1 \geq i_2 \geq \ldots \geq i_k \geq 1\}$$

is a basis for $\mathcal{U}(L)$.

From PBW we see that ι' is *injective*. Then for a representation of $\mathcal{U}(L)$, restriction to the subspace $L = \iota(L)$ furnishes a representation of L. In this way, representations of L and $\mathcal{U}(L)$ determine each other in a canonical fashion - a statement that can be better stated using categories of modules (omitted).

The Lie algebra L has a *triangular decomposition* if it decomposes as

$$L = L^+ \oplus L^0 \oplus L^-$$

such that $L^\pm, L^0$ are Lie subalgebras, and the bracket satisfies

$$[L^+L^-] \subseteq L^0, \quad [L^\pm L^0] \subseteq L^\pm.$$

Use of PBW and an appropriate choice of (ordered) basis leads to an identification

$$\mathcal{U}(L) = \mathcal{U}(L^-) \otimes \mathcal{U}(L^0) \otimes \mathcal{U}(L^+).$$

Noting that $L^0 \oplus L^+ \subseteq L$ is a Lie subalgebra, let $\pi : L^0 \oplus L^+ \longrightarrow \mathrm{End}(V)$ be a representation. The *induced module* is

$$\mathrm{Ind}(V) = \mathrm{Ind}_{\mathcal{U}(L^0 \oplus L^+)}^{\mathcal{U}(L)} V := \mathcal{U}(L) \otimes_{\mathcal{U}(L^0 \oplus L^+)} V = \mathcal{U}(L^-) \otimes V.$$

It is a $\mathcal{U}(L)$-module, hence also an L-module upon restriction.

Exercise: Show that the following Lie algebras have natural triangular decompositions:

(a) Heisenberg algebra $\hat{A}$ with

$$\hat{A}^+ = \oplus_{n>0}\mathbb{C}a \otimes t^n, \ \hat{A}^- = \oplus_{n<0}\mathbb{C}a \otimes t^n, \ \hat{A}^0 = \mathbb{C}a \otimes t^0 \oplus \mathbb{C}K.$$

(b) Virasoro algebra Vir with

$$Vir^+ = \oplus_{n>0}\mathbb{C}L_n, \ Vir^- = \oplus_{n<0}\mathbb{C}L_n, \ Vir^0 = \mathbb{C}L_0 \oplus \mathbb{C}K.$$

(c) Finite-dimensional semisimple Lie algebras (equipped with a choice of Cartan subalgebra and root system) with

$$L^\pm = \{\text{positive/negative root spaces}\}, L^0 = \{\text{Cartan subalgebra}\}.$$

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