3/11/18

Quantum Characteristic Classes

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Position, Momentum, Locality, & Stability

Work by: Wigner, Bellissard, Kitaev, Quinn, Pederson, Ree

Suppose we have $f: \mathbb{R} \to \mathbb{C}$

$$\hat{f}(K) = \int e^{iKx} f(x)\,dx$$

$$\check{f}(y,K) = \sum_{x \in \mathbb{Z}} e^{iKx} f(x+y), \quad \begin{matrix} y \in [0,1) \\ K \in [0,2\pi) \end{matrix}$$

Notice

$$\check{f}(1,K) = \sum_{x \in \mathbb{Z}} e^{iK(x-1)} f(x)$$

$$= e^{-iK} \check{f}(0,K)$$

Section of line bundle $c_1 = 1$

U_{ij} , unitary matrix

$U_{ij}^* U_{ik} = \delta_{ij} \quad i,j \in \mathbb{Z}$

local: $U_{ij} = 0$, if $|i-j| > \ell$, ℓ some constant

$$f_{jk} = |u_{jk}|^2 = |u_{kj}|^2$$

$$f(u) = \sum_{i<c\leq j} f_{ij}$$, index of c

<u>the flow</u>

<u>Prove</u> <u>$f(u) \in \mathbb{Z}$</u>

Equivalent definition of $f(u)$

Let $\pi = \text{proj } [0,1,2,\dots)$. Then,

$$f(u) = \mathrm{tr}\left(U^\dagger \pi U (1-\pi)\right) - \mathrm{tr}\left(U^\dagger (1-\pi) U \pi\right)$$

$$= \mathrm{tr}(\underbrace{U^\dagger \pi U - \pi}_{0 \text{ unless } |i|,|j|<\ell})$$

$$= \mathrm{tr}\left(T U^\dagger \pi U T - T \pi T\right) \in \mathbb{Z}$$

$T = \text{proj}$

$U_{s,\lambda,t,v}$, depends on $t-s, \lambda, v$

$$\tilde{U}_{\lambda,v}(q) = \sum e^{iqt} U_{0,\lambda,t,v}$$

$$f(U) = \int_{q=0}^{2\pi} \mathrm{tr}\, \tilde{U}^{+} \frac{d\tilde{U}}{dq} = \int_{q=0}^{2\pi} d\left(\log \det \tilde{U}(q)\right)$$

$$= \mathrm{tr}\,(\log)$$

————————

Index space $\mathbb{Z} \oplus \mathbb{Z}$

P_{ij} ; $P^2 = P$; $P^{+} = P$, $P_{ij} = 0$, unless $\|i-j\| \le \ell$

<u>2-chain</u>

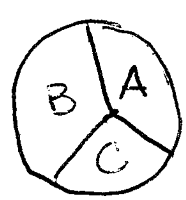
$$h_{ijk} = 12\pi i \left(P_{ij} P_{jk} P_{ki} - P_{ik} P_{kj} P_{ji} \right)$$

* this is a closed 2-chain

i.e. $(\partial h)_{jk} := \sum_i h_{ijk} = 0$

<u>Chern:</u>

$$v(P) = \sum_{i \in A} \sum_{j \in B} \sum_{k \in C} h_{ijk}$$



this is an integer

$$\nu(P) = 12\pi i \left[\operatorname{tr}(APBPCP) - \operatorname{tr}(CPBPAP)\right]$$

$A+B+C=1$

$$= \operatorname{tr} 4\pi i \,[PAP,\ PBP]$$

$$= \operatorname{tr} 2\pi i \,[P\Pi^{X}P,\ P\Pi^{Y}P]$$

$$= \operatorname{tr}_c\, 2\pi i\,[PXP,\ PYP]$$

$$= \operatorname{tr}_c\, 2\pi i\, P\big[[X,P],\,[Y,P]\big]$$

$\downarrow$ momentum

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \operatorname{tr} \tilde{P}\left(\frac{\partial \tilde{P}}{\partial q_x}\frac{\partial \tilde{P}}{\partial q_y} - \frac{\partial \tilde{P}}{\partial q_y}\frac{\partial \tilde{P}}{\partial q_x}\right) dq_x\, dq_y$$

$$= \frac{1}{2\pi i}\int \operatorname{tr} \tilde{P} \wedge d\tilde{P} \wedge d\tilde{P}$$

$$= c_1$$

Relation between ν and f

Start with P

$Q^x = P\pi^x P$

$Q^y = P\pi^y P$

$U^x = e^{2\pi i Q^y}$

$$f(U) = \mathrm{tr}\left(e^{-2\pi i Q^y} Q^x e^{2\pi i Q^y} - Q^x\right)$$

$$\overset{\text{f.t.c.}}{=} \int_0^{2\pi} \mathrm{tr} \frac{d}{dq}\left(e^{-i\phi Q^y} Q^x e^{i\phi Q^y}\right) dq$$

$$= 2\pi i \, \mathrm{tr} [Q^x, Q^y]$$

$$= 2\pi i \; \mathrm{tr} [P\pi^x P, P\pi^y P] = \nu(P)$$

Quantum-char. classes (K theory $\mathbb{Q}$) K. Taeu, Wigner, Bellissard

basic things: position momentum locality stability [Joke]

1. $f(x): \mathbb{R} \to \mathbb{C}$ $\qquad \hat{f}(k) = \int dx\, e^{ixk} f(x)$

$\check{f}(y,k) = \sum_{x \in \mathbb{Z}} e^{ikx} f(x+y)$, $y \in [0,1)$, $k \in [0, 2\pi)$, $\underbrace{\check{f}(1,k) = \sum e^{ik(x-1)} f(x) = e^{-ik} \check{f}}_{C_1}$

2. U_{ij} $\quad i,j \in \mathbb{Z}$, $U^*_{ij} U_{ik} = \delta_{jk}$, $\quad U_{ij} = 0$ if $|i-j| > \ell$

flow $k \to j$ $\qquad f_{jk} = |U_{jk}|^2 - |U_{kj}|^2 \qquad (\sum_j f_{jk} = 0)$

Def $\qquad f(U) = \sum_{k < c \le j} f_{jk} \qquad$ (ind. of c)

Prove integrality $\qquad \Pi = \text{proj}(0, 1, \dots,)$ $\qquad$ careful!

$f(U) = \mathrm{tr}\,(U^\dagger \Pi U (1-\Pi) - U^\dagger (1-\Pi) U \Pi) = \mathrm{tr}\,(\overbrace{U^\dagger \Pi U - \Pi}^{\Lambda})$ $\qquad \Lambda_{ij} = 0$ if either $|i| \text{ or } |j| > \ell$

$T = \Pi^{[-\ell, \ell]}$ $\qquad \Pi, \Lambda$ and $\therefore$ $U^\dagger \Pi U$ commute with T

$\therefore$ $TU^\dagger \Pi U T$ and $T \Pi T$ are still orthog. proj. $\Rightarrow$ integer $= \underline{\underline{\mathrm{tr}\, U^\dagger [\Pi, U]}}$

3. Beautiful formula for $f(U)$ in trans inv. case.

$U_{s,\lambda; t, \nu}$ $\quad$ depends only on $t-s, \lambda, \nu$

$f(U) = \mathrm{tr}_{\text{cell}}\, U^\dagger [X, U]$

$= \frac{i}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}\, \tilde{U}^\dagger \frac{d\tilde{U}}{dq} dq = \frac{i}{2\pi} \int_{-\pi}^{\pi} d(\log \det \tilde{U}(q))$

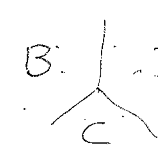
formuli:

1. $X = s\, \delta_{s,t} \delta_{\lambda,\nu}$
2. $\tilde{U}(q)_{\lambda,\nu} = \sum e^{iqt} U_{0\lambda t \nu}$
3. $\mathrm{tr}\, A = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}\, \tilde{A}\, dq$
4. AB [illegible]

4. P a local projector $\quad$ 2-D index set, say $\mathbb{Z} \times \mathbb{Z}$

$P^2 = P$ $\Downarrow$

2-chain (closed) $\quad h_{ijk} = 12\pi i\,(P_{ij} P_{jk} P_{ki} - P_{ik} P_{kj} P_{ji})$ $\qquad$ real $P = P^\dagger$ + reverses $\qquad (\partial h)_{jk} = \sum_i h_{ijk} = 0$



Defs $\quad \nu(P) = \sum_{j \in A} \sum_{k \in B} \sum_{\ell \in C} h_{jk\ell}$ $\qquad C = 1 - A - B$

$= 12\pi i \quad \mathrm{tr}\,(APBPCP) - \mathrm{tr}\,(CPBPAP)$

$= 4\pi i \quad \mathrm{tr}\,[PAP, PBP]$

$= 2\pi i \quad \mathrm{tr}\,[P\Pi^x P, P\Pi^y P] = 2\pi i\, \mathrm{tr}_c [PXP, PYP]$

✓ $= \frac{1}{2\pi i} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{tr}\, \tilde{P} \left(\frac{\partial \tilde{P}}{\partial q_x} \frac{\partial \tilde{P}}{\partial q_y} - \frac{\partial \tilde{P}}{\partial q_y} \frac{\partial \tilde{P}}{\partial q_x} \right) dq_x\, dq_y = \frac{1}{2\pi i} \int_{T^2} \mathrm{tr}\, \tilde{P} (d\tilde{P} \wedge d\tilde{P})$

$\tilde{P} = \sum_{\gamma} e^{i\gamma q} P_{0\lambda \gamma \nu}$

[illegible]

5, Relation ν to f (Chern, flow)

start with P define $Q^x = P\Pi^x P$, $Q^y = \cdots$, $U^x = e^{2\pi i Q^y}$, $U^y = \cdots$ quasi 1-D

$f(U^x)$ = flow on image P (identity $\mathbb{1}$)

$$= \operatorname{tr}\left(e^{-2\pi i Q^y} \underset{Q^x}{\Pi^x} e^{2\pi i Q^y} - \underset{Q^x}{\cancel{\Pi^x}} \right)$$

$$\overset{F.T.C.}{=} \int_0^{2\pi} \operatorname{tr} \frac{d}{d\phi} \left(e^{-i\phi Q^y} Q^x e^{i\phi Q^y} \right) d\phi$$

$$= 2\pi i \ \operatorname{tr}[Q^x, Q^y] = 2\pi i \ \operatorname{tr}[P\Pi^x P, P\Pi^y P] = \nu(P)$$

1. Ex 2D lattice Hamiltonian $H_{8\times 2N} \xrightarrow{F.T.} \hat{H}(q)$ "Chern ..."

[This is a very interesting "line" in $\mathbb{C}^2 \otimes \ldots_N \otimes \mathbb{C}^2$

2 In topology we are used to a democracy of lines, k-planes

not so in physics/controlled ... few lines (k-planes) have "good" local descriptions.

3 ... new "char. classes" local lines ... $\mathbb{Z}+\mathbb{Z}$ ...

connected space ... $\to (P)$ $\Omega^d KU_d = KU_d^d \cdot KU$

Interacting "Good" = loc, gapped, stable (periodic)

4 Coarse R^n (conj): no good k-plane, $k > 1$. (translation ...)

loc. gapped stable

5 Coarse Σ^2: FQHE and other QFTs

Π_K ...

$U \simeq$ ... BU U ...

6 ... good k-planes over M^d

$\pi_0 U^1 = \pi_{-1} U \overset{B?}{=} \pi_1 U$

$\pi_1 KU^2$ ... KU ...

$e^{2\pi i\, c/2}$ ...

... 1 2 3 4 ...

K_L

$[X, K]$ ...