Twisted Alexander polynomials and fibrations of 3-mnfds

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Conjecture 1:
Let N^3 closed, $S^1 \times N$ is symplectic
$\Rightarrow$ N fibers over S^1.

($\phi \in H^1(N;\mathbb{Z}) = [N, S^1]$ is fibered
(N, ϕ) fibers)

Starting point: $N = S^1 \times S^2$ or irreducible

From now on, we assume N irreducible.

Thm (Friedl, Vidussi '05):
If $(S^1 \times N, \omega)$ is an integral symplectic mnfd, $\phi \in H^1(N;\mathbb{Z})$ Künneth component of $[\omega]$, then (N, ϕ) satisfies:
Condition $(*)$: $\forall\ \tilde{\pi} \trianglelefteq \pi_1(N)$, $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$ satisfies
1. $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$ is monic
2. $\deg \Delta^{\pi/\tilde{\pi}}_{N,\phi} = [\pi : \tilde{\pi}]\, \|\phi\|_T + 2 \operatorname{div} \phi|_{\tilde{\pi}}$

Conjecture 2: If (N, ϕ) satisfies condition $(*)$, then (N, ϕ) fibers.

Main theorem: Conjecture 2 holds.

Strategy for proof:

(N, ϕ), ϕ primitive
$\Sigma \in PD[\phi]$, connected & $\|\cdot\|_T$-minimizing

$M = N \setminus \nu\Sigma$ $\quad i_\pm : \pi_1(\Sigma) \hookrightarrow \pi_1(M)$
$\cong$ iff (N, ϕ) fibers

Step 1: If (N, ϕ) satisfies condition $(*)$, then $i_\pm$ induces an isomorphism of prosolvable completions.

Step 2: If $i_\pm : \pi_1(\Sigma) \longrightarrow \pi_1(M)$ induces an iso. of prosolvable completions and $\pi_1(M)$ is residually finite solvable, then $M = \Sigma \times I$.

Preliminaries:
V, $\mathbb{Z}$-module carrying a rep
$\alpha : \pi_1(N) \longrightarrow \mathrm{Aut}_{\mathbb{Z}} V \Rightarrow$ twisted
$\pi_1(N) = \pi$

Alexander module of (N, ϕ, α) $\quad H_1(\pi, V[t, t^{-1}])$

$$\Delta^{\alpha}_{N,\phi} := \operatorname{ord} \operatorname{Tors}_{\mathbb{Z}[t,t^{-1}]} H_i(\pi, V[t^{\pm 1}]) \in \mathbb{Z}[t,t^{-1}]$$

Def: a map $\phi : A \longrightarrow B$ btwn fin. gen'd groups induces an isomorphism of prosolvable completions if $\forall$ G finite solvable $\phi^* : \mathrm{Hom}(B,G) \xrightarrow{\cong} \mathrm{Hom}(A,G)$.

$M = N \setminus \nu\Sigma$; $A = \pi_1(\Sigma)$, $B = \pi_1(M)$

$i_{\pm} : A \longrightarrow B$, $\pi = \pi_1(N)$

Lemma: Assume (N,ϕ) satisfies condition $(*)$, then $\forall$ $\alpha : \pi \to G$, finite, $i_{\pm} : H_1(A; \mathbb{Z}[G]) \to H_1(B, \mathbb{Z}[G])$ is an isomorphism.

$$H_1(A, \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \xrightarrow{t} H_1(B, \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$$

$$\to H_1(\pi, \mathbb{Z}[G][t^{\pm 1}]) \to 0$$

$\Rightarrow \Delta^{\alpha}_{N,\phi} = \det(t$

$\overset{2}{\Rightarrow} \deg \Delta_{N,\phi} = h_1(A, \mathbb{Z}[G])$

$\overset{1}{\Rightarrow} \det i_{\pm} = \pm 1$ & $i_{\pm}$ is an isomorphism

Thm: Let (N, ϕ) be as above. $i_\pm : A \to B$ induces an isomorphism of prosolvable completions.

Pf: (sketch)

$S(n)$: for any solvable group S of $\ell(S) \leq n$

$$i_\pm^* : \mathrm{Hom}(B, S) \xrightarrow{\cong} \mathrm{Hom}(A, S)$$

$H(n)$: $\forall\ \rho : B \to S$ solvable of $\ell(S) \leq n$,
$$i_\pm : H_1(A; \mathbb{Z}[S]) \to H_1(B, \mathbb{Z}[S])$$

Prop A: If $H(n)$ & $S(n)$ hold, so does $S(n+1)$.

Prop B: If $S(n)$ holds, so does $H(n)$.

"Proof" of A:

$S(n+1)$, G, $\ell(G) = n+1$)

$\exists\ \Phi : \mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G)$ inverting 2.

$$\begin{tikzcd}
1 \arrow[r] & G^{(n)} \arrow[r] & G \arrow[r] & S \arrow[r] & 1 \\
& A/\mathrm{Ker}\,\alpha \arrow[ur] & & & \\
& & A \arrow[ul] \arrow[uu, "\alpha"] \arrow[uur, "\alpha'"] \arrow[r] & B \arrow[uul] \arrow[uu, "\beta'"] &
\end{tikzcd}$$

$\alpha(\mathrm{Ker}\,\alpha') \subseteq G^{(n)}$

$$\begin{tikzcd}
 & {[\mathrm{Ker}\,\alpha', \mathrm{Ker}\,\alpha']} \arrow[d] & {[\mathrm{Ker}\,\alpha', \mathrm{Ker}\,\alpha']} \arrow[d] & & \\
1 \arrow[r] & \mathrm{Ker}\,\alpha' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & S \arrow[r] \arrow[dr, "\cong"] & 1 \\
0 \arrow[r] & H_1(A; \mathbb{Z}[S]) \arrow[r] & A/[\mathrm{Ker}\,\alpha', \mathrm{Ker}\,\alpha'] \arrow[rr] & & S \arrow[r] & 1 \\
 & H_1(B; \mathbb{Z}[S]) \arrow[r] \arrow[u, "\cong"] & B/[\mathrm{Ker}\,\beta', \mathrm{Ker}\,\beta'] \arrow[rr] \arrow[u, "\cong"] & & S \arrow[u, "\cong"] &
\end{tikzcd}$$

"Proof" of B

$\beta : B \to S$ "extend" β

Def: $B(S) = \bigcap_{\gamma \in \mathrm{Hom}(B,S)} \mathrm{Ker}\,\gamma$

$$\begin{tikzcd}
\text{Claim: } \exists & 1 \arrow[r] & B/B(S) \arrow[r] & \mathbb{Z}_K \ltimes B/B(S) \arrow[r] & \mathbb{Z}_K \arrow[d] \\
 & & B \arrow[u] & \pi \arrow[u] & 1
\end{tikzcd}$$

$$\Rightarrow i_* : H_1(A; \mathbb{Z}[B/B(S)]) \xrightarrow{\cong} H_1(B; \mathbb{Z}[B/B(S)])$$

Def: a group π is called RFRS if $\exists$ a filtration $\pi = \pi_0 \supset \pi_1 \supset \cdots \supset \pi_i$ s.t. ① $\bigcap_i \pi_i = \{1\}$

② $\pi_i \trianglelefteq \pi_0$

③ $\pi_i \to \pi_i / \pi_{i+1}$ factors through

$$\pi_i \to H_1(\pi_i, \mathbb{Z}) / \mathrm{Tor}$$

$M = N \setminus \nu\Sigma \qquad DM = M \cup M$

$r : DM \to M$

Thm (Agol): If $\pi_1(M)$ is RFRS, $\exists$ $\alpha : \pi_1(M) \to S$, finite solvable, s.t. in the cover $p : \widetilde{DM} \to DM$ determined by $\alpha \circ r_* \pi_1(M) \to S$, the class $p^{-1}\Sigma$ is in the closure of a fibered cone of $H_2(\widetilde{DM}; \mathbb{Z})$.

Thm: Assume that

1. $\pi_1(\Sigma) \xrightarrow{i_*} \pi_1(M)$ induces an iso. of prosolv. completions
2. $\pi_1 M$ is residually finite solvable, then

(N, ϕ) is fibered.

Claim: $\pi_1 M$ is RFRS

<u>Lemma</u>: let M irred. mnfd with incompressible bdy $\Sigma^+ \cup \Sigma^-$, $DM = M \cup M$
1. $\pi_1(\Sigma^\pm) \longrightarrow \pi_1(M)$ induces an iso. of prosolvable completions.
2. the class of $H^1(DM; \mathbb{Z})$ represented by $[\Sigma^-]$ is the closure of a fibered cone

N has linear π_1 : easy

In general: break N along JSJ tori, N_i

$$\begin{array}{ccc} \pi_1(\Sigma) & \hookrightarrow & \pi_1(M) \\ \downarrow & & \\ \pi_1(\Sigma \cap N_i) & \hookrightarrow & \pi_1(M \cap N_i) \end{array}$$

Twisted Alexander polynomials & fibered 3-mflds

1. Introduction

Purpose: discuss the conjecture

Conjecture 1: let N be a closed 3-mfld; $S^1 \times N$ admits a symplectic structure iff N fibers over S^1

(N fibers over S^1: $\exists\, \phi \in H^1(N;\mathbb{Z}) = [N, S^1]$ represented by a fibration; (N,ϕ) is a fibered pair.)

$\Leftarrow$ Thurston, $\Rightarrow$ mounting evidence, built on results & ideas of Kronheimer, Taubes

Our approach: use constraints on TAP's of N arising from symplectic condition to show N fibers

Starting point (McCarthy, '01): N prime.

Now on, N irreducible

Theorem (Friedl-V, '05) If $(S^1 \times N, \omega)$ integral symplectic mfld, $\phi \in H^1(N;\mathbb{Z})$ künneth component of $[\omega]$, then (N,ϕ) satisfies

<u>Condition (*)</u>. $\forall\ \tilde{\pi}$ s.f.i. $\pi := \pi_1(N)$, the TAP $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$

satisfies: <u>1. $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is monic</u>;

<u>2. $\deg(\Delta_{N,\phi}^{\pi/\tilde{\pi}}) = [\pi : \tilde{\pi}] \|\phi\|_T + 2\,\mathrm{div}\, \phi|_{\tilde{\pi}}$</u>

<u>Tools of the proof</u>:

1. Taubes' constraints on SW invs of $S^1 \times N$
2. Meng-Taubes relation $SW - \Delta_N$
3. Donaldson Theorem on PD$[\omega]$
4. Kronheimer's refined adjunction inequality
5. Relation TAPs of N & A.P. of covers of N □

<u>Evidence to Conjecture 1</u>: If (N,ϕ) fibers, (N,ϕ) satisfies

Condition (*). leads to formulate

<u>Conjecture 2</u>. let (N,ϕ), $\phi \neq 0 \in H^1(N;\mathbb{Z})$ satisfy Condition (*)

then (N,ϕ) fibers

<u>Conjecture 2 $\Rightarrow$ Conjecture 1</u>

Theorem (Friedl - V, '06): Conjecture 2 holds true when either

1 N has vanishing $\|\ \|_T$; 2. N is a graph manifold & π is LERF

2 Main Theorem and strategy of the proof

2 **Main theorem** (Friedl–V. '08) Conjecture 2 holds true

Remark Kutluhan–Taubes: if $S^1 \times N$ is symplectic, N has SWF of a fibered 3-mfld

Strategy of the proof: can assume ϕ primitive, $\exists$ connected $\|\ \|_T$-minimizing $\Sigma \in PD(\phi)$

Denote $M = N \setminus \nu\Sigma$. Stallings: Σ fiber $\Leftrightarrow \iota_\pm : \pi_1(\Sigma) \xrightarrow{\cong} \pi_1(M)$

Step 1: if (N, ϕ) satisfies Condition $(*)$, $\iota_\pm : \pi_1(\Sigma) \to \pi_1(M)$ induces iso of prosolvable completions

Appl **Step 2**: if N is a 3-mfld with empty or toroidal boundary, $\Sigma \hookrightarrow N$ satisfies 1. $\iota_\pm : \pi_1(\Sigma) \to \pi_1(M)$ induces iso of prosolvable completions; 2. $\pi_1(M)$ is residually f. solvable then $M = \Sigma \times I$

Step 3: if π is virtually residually f. solvable (e.g. N hyperbolic), basically done; otherwise work with pieces of JSJ decomposition

3. Preliminaries

Given a $\mathbb{Z}$-module V that carries a representation α of $\pi = \pi_1(N)$, we can associate to (N, ϕ) the <u>twisted Alexander module</u> <u>$H_1(\pi, V[t^{\pm 1}])$</u>;

when the module is $\mathbb{Z}[t^{\pm 1}]$-torsion, the TAP

<u>$\Delta^{\alpha}_{N,\phi} = \operatorname{ord} \operatorname{Tors} H_1(\pi, V[t^{\pm 1}])$</u> $\in \mathbb{Z}[t^{\pm 1}]$.

Given <u>$\tilde{\pi} \leq_f \pi$</u>, we can consider the permutation module <u>$\mathbb{Z}[\pi/\tilde{\pi}]$</u>, and the corresponding TAP <u>$\Delta^{\pi/\tilde{\pi}}_{N\phi}$</u> related to the AP of the $\tilde{\pi}$-cover of N

Let $\mathcal{C}$ a variety of finite groups (set of groups closed under quotient, subgroup, product).

Let A be a f.g. group. The <u>completion $\hat{A}_{\mathcal{C}}$</u> of A is the solution to the universal problem

$$\begin{tikzcd} & \hat{A}_{\mathcal{C}} \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & G \end{tikzcd}$$

for all maps $A \to G \in \mathcal{C}$. It satisfies all right functorial properties, in particular given $\varphi : A \to B$

$\exists$
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ \hat{A}_{\mathcal{C}} \arrow[r, "\hat{\varphi}"] & \hat{B}_{\mathcal{C}} \end{tikzcd}$$

We will use the following characterizing property:

Claim: $\varphi : A \to B$ induces an isomorphism of pro-$\mathcal{C}$ completions iff $\forall G \in \mathcal{C}$, $\varphi^* : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$ is a bijection.

We will be interested in $\mathcal{C}$ = finite soluble groups

4. Constraints from Condition (*)

Recall the definition of $M = N \setminus \nu\Sigma$; denote $A = \pi_1(\Sigma)$, $B = \pi_1(M)$.

Lemma Let (N, ϕ) satisfy Condition (*), then $\forall\, \alpha : \pi \to G$ finite, $\iota_\pm : H_1(A; \mathbb{Z}[G]) \xrightarrow{\cong} H_1(B; \mathbb{Z}[G])$

Proof (sketch): we have a resolution of $H_1(\pi; \mathbb{Z}[G][t^{\pm 1}])$ by free $\mathbb{Z}[t^{\pm 1}]$-module of equal rank:

$$H_1(A; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{t\iota_+ - \iota_-} H_1(B; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \to H_1(\pi; \mathbb{Z}[G][t^{\pm 1}]) \to 0,$$

hence $\Delta^\alpha_{N,\phi} = \det(t\iota_+ - \iota_-)$; condition 2 in (*) implies $\deg \det(t\iota_+ - \iota_-) = \operatorname{rk} H_1(A; \mathbb{Z}[G])$, and so the leading coefficient is $\det \iota_+ = \pm 1$.

It follows ι_+ (and by symmetry, ι_-), is an isomorphism.

Theorem Let (N, ϕ) as above: then $\iota_\pm : A \to B$ induces an isomorphism of prosolvable completions

Proof (sketch) Let $S(n)$ be the statement that for any solvable group S of $\ell(S) \leq n$ and $\iota = \iota_\pm$,

$i_\pm^*: \mathrm{Hom}(B,S) \to \mathrm{Hom}(A,S)$ is a bijection;

let $H(n)$ be the statement that for any $\beta: B \to S$ solvable, $\ell(S) \le n$, $i_\pm: H_1(A;\mathbb{Z}[S]) \cong H_1(B;\mathbb{Z}[S])$.

$S(0)$ holds by fiat, $H(0)$ follows from the Lemma.

The theorem is therefore a consequence of

<u>Proposition A</u>: if $H(n)$ & $S(n)$ hold, so does $S(n+1)$

Proposition B: if $S(n)$ holds, so does $H(n)$

<u>Proof of Proposition A</u>: given G, $\ell(G) = n+1$

need to show $\exists\, \Phi: \mathrm{Hom}(A,G) \to \mathrm{Hom}(B,G)$

inverting $i^*: \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$

We have

$$\begin{tikzcd}
1 \arrow[r] & \underset{\text{abelian}}{G^{(n)}} \arrow[r] & G \arrow[r] & S \arrow[r] & 1 \\
& & A \arrow[u, "\alpha"] \arrow[ur, dashed, "\alpha'"] \arrow[r] & B \arrow[u, "\beta' \text{ by } S(n)"'] &
\end{tikzcd}$$

$\alpha(\ker\alpha') \subseteq G^{(n)}$, hence $\alpha(\ker\alpha'^{(1)}) = \{1\} \in G$,

so we have a factorization

$$\begin{tikzcd}
G \arrow[rr] & & S \\
& A/\ker\alpha'^{(1)} \arrow[ul] & \\
A \arrow[uu] \arrow[ur] \arrow[rr] & & B \arrow[uu, "\beta'"']
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
& \ker\alpha'^{(1)} \arrow[r] \arrow[d] & \ker\alpha'^{(1)} \arrow[d] & \\
1 \arrow[r] & \ker\alpha' \arrow[r] \arrow[d] & A \arrow[r, "\alpha'"] \arrow[d] & S \arrow[d, "\cong"] \\
& H_1(A;\mathbb{Z}[S]) \arrow[r] & A/\ker\alpha'^{(1)} \arrow[r] & S
\end{tikzcd}$$

$S(n)$ & $H(n)$ guarantee that the corresponding diagram for $\beta' : B \to S$ has the same bottom row;

$\exists$ an isomorphism $B/\ker\beta'^{(1)} \longrightarrow A/\ker\alpha'^{(1)}$ that we use to define γ, hence Φ $\square$

Proof of Proposition B: given $\beta : B \to S$, we want to "extend" $\beta : B \to S$ to π. Define $B(S) = \bigcap_{\gamma \in \mathrm{Hom}(B,S)} \ker\gamma$.

Step 1: $\exists$ a diagram

$$\begin{tikzcd}
1 \arrow[r] & B/B(S) \arrow[r] & \mathbb{Z}_k \ltimes B/B(S) \arrow[r] & \mathbb{Z}_k \arrow[r] & 1 \\
& B \arrow[u] \arrow[r] & \pi \arrow[u] & &
\end{tikzcd}$$

Step 2: Condition $(*)$ implies $H_1(A;\mathbb{Z}[B/B(S)]) \to$

$\to H_1(B;\mathbb{Z}[B/B(S)])$; out of that the isomorphism follows. $\square$

5. A product criterion

Def: a group π is called RFRS if $\exists$ a filtration $\pi = \pi_0 \supset \pi_1 \supset \dots \supset \pi_i$ s.t.

1. $\bigcap_i \pi_i = \{1\}$

2 $\pi_i \triangleleft_{f_i} \pi_0$

3. $\pi_i \longrightarrow \pi_i / \pi_{i+1}$ factors through $\pi_i \longrightarrow H_1(\pi_i; \mathbb{Z})/_{Tor}$

Let $M = N \setminus \nu\Sigma$, and define the double $DM = M \cup M$, with the folding map $r: DM \longrightarrow M$

Theorem (Agol) with the notation above, if $\pi_1(M)$ is RFRS $\exists$ an epimorphism $\alpha: \pi_1(M) \longrightarrow S$ finite solvable s.t. in the cover $p: \widetilde{DM} \longrightarrow DM$ determined by $\alpha \circ r_*: \pi_1(\widetilde{DM}) \longrightarrow S$ the class of $p^{-1}\Sigma^-$ lies in the closure of a fibered cone.

Theorem with the notation above, assume that

1. $i_*: \pi_1(\Sigma) \longrightarrow \pi_1(M)$ induces an isomorphism of prosolvable completions; 2. $\pi_1(M)$ is residually finite solvable

Then (N, ϕ) fibers with fiber Σ

Proof. First, we claim that $\pi_1(M)$ is RFRS.

In fact, $\exists$ a filtration that satisfies 1, & 2. and

3! $\pi_i \to \pi_i / \pi_{i+1}$ factorizes through $\pi_i \to H_1(\pi_i; \mathbb{Z})$;

but $H_1(\pi_i; \mathbb{Z}) = H_1(\pi_1(M); \mathbb{Z}[\pi_1(M)/\pi_i])$, torsion free.

by 1. Now we can work on $\widetilde{DM}$, it is not

difficult to see that $\tilde{\Sigma}^{\pm} = p^{-1}(\Sigma^{\pm})$ is connected,

and $\pi_1(\tilde{\Sigma}^{\pm}) \to \pi_1(\tilde{M})$ induces an isomorphism

of prosolvable completions. Also, $M = \Sigma \times I$ iff

$\tilde{M} = \tilde{\Sigma} \times I$, which in turn is equivalent to $\tilde{\Sigma}$

being a fiber of $\widetilde{DM} \to S^1$. Therefore the following

lemma proves the theorem

Lemma let M be a irreducible 3-mfld with incompressible

boundary, and assume

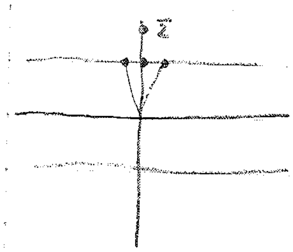
1. $\pi_1(\Sigma^{\pm}) \to \pi_1(M)$ induces an isomorphism of prosolvable

completion; 2. The class of $H^1(DM; \mathbb{Z})$

represented by Σ^- lies in the closure of a fibered cone then Σ^- is actually a fiber

<u>Proof</u> (sketch) (1) implies that $\exists$ an isomorphism <u>$f: H_2(\Sigma \times S^1; \mathbb{R}) \to H_2(DH; \mathbb{R})$</u> with $f([\Sigma]) = [\Sigma^-]$ preserving the Alexander norm; hence $[\Sigma]$ lies over a top dimensional face of the Alexander norm ball

(2) and the existence of an involution implies that $[\Sigma^-]$ must be in a top dimensional face of the Thurston norm ball as well.



6. Completion of the proof

If N is a 3-mfld whose group is virtually residually finite solvable (e.g. hyperbolic 3-mfld), $\exists$ a cover $p: \tilde{N} \to N$ whose group is residually finite solvable.

Now (N, ϕ) is fibered iff $(\tilde{N}, p^*\phi)$ is, and if (N, ϕ) satisfies condition $(*)$, so does $(\tilde{N}, p^*\phi)$.

For a general 3-mfld, we use the following result

Claim: let N be a irreducible 3-mfld, $\exists\, p: \tilde{N} \to N$ s.t. the group of each JSJ component is residually finite solvable. Working on the cover, denoting $\tilde{N}_i$ the components of the JSJ decomposition, the isomorphism of prosolvable completions for $\pi_1(\tilde{\Sigma}) \to \pi_1(\tilde{M})$ induces isomorphism of prosolvable completions for the single JSJ components $\pi_1(\tilde{\Sigma} \cap \tilde{N}_i) \to \pi_1(\tilde{M} \cap \tilde{N}_i)$.