"An Algebraic Structure"

Projectivization, w-Knots, Kashiwara–Vergne and Alekseev–Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm. Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y) = e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- ◦ The Alekseev-Torossian equations in $\mathcal{U}(\text{sder}_n)$ and $\mathcal{U}(\text{tder}_n)$. sder $\leftrightarrow$ tree-level $\mathcal{A}$, tder $\leftrightarrow$ more

$$F \in \mathcal{U}(\text{tder}_2);\quad F^{-1}e(x+y)F = e(x)e(y) \quad\Longleftrightarrow\quad F \in \text{Sol}_0$$

$$\Phi = \Phi_F := (F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23} \in \mathcal{U}(\text{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4} = \Phi^{12,3,4}\Phi^{1,2,34} \qquad \text{"the pentagon"}$$

$t = \frac{1}{2}(y,x) \in \text{sder}_2$ satisfies $4T$ and $r = (y,0) \in \text{tder}_2$ satisfies $6T$

$$R := e(r) \text{ satisfies Yang-Baxter: } R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

$$\text{also } R^{12,3} = R^{13}R^{23} \text{ and } F^{23}R^{1,23}(F^{23})^{-1} = R^{12}R^{13}$$

$$\tau(F) := RF^{21}e(-t) \text{ is an involution}, \Phi_{\tau(F)} = (\Phi_F^{321})^{-1}$$

$$\text{Sol}_0^\tau := \{F\colon \tau(F) = F\} \text{ is non-empty; for } F \in \text{Sol}_0^\tau,$$

$$e(t^{13}+t^{23}) = \Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13}) = (\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$


Alekseev

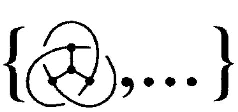
This is just a part of the Alekseev–Torossian work!

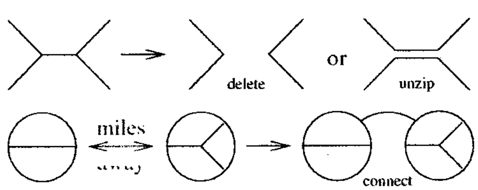

Torossian

So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

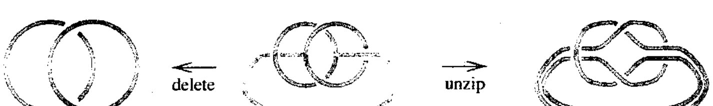
Knotted Trivalent Graphs

$\mathcal{O}(\triangle) = \{$  $, \dots\}$



Theorem. KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.

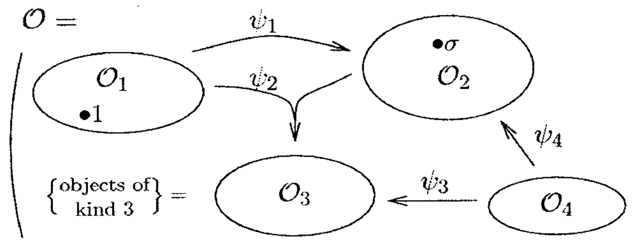
Theorem. $Z(\triangle)$ is equivalent to an associator Φ.



Algebraic Knot Theory

Theorem. $\{\text{ribbon knots}\} \sim \{u\gamma\colon \gamma \in \mathcal{O}(\circ\!\!-\!\!\circ),\ d\gamma = \bigcirc\bigcirc\}$.

Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

$\mathcal{O} =$ 

- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$. The augmentation "ideal":

$$I = I_{\mathcal{O}} := \left\{\begin{matrix}\text{formal differences of ob-}\\ \text{jects "of the same kind"}\end{matrix}\right\}.$$

$$\text{Then } I^n := \left\{\begin{matrix}\text{all outputs of algebraic}\\ \text{expressions at least } n \text{ of}\\ \text{whose inputs are in } I\end{matrix}\right\}, \text{ and}$$

$$\text{proj}\,\mathcal{O} := \bigoplus_{n\geq 0} I^n/I^{n+1} \quad \left(\begin{matrix}\text{has same kinds and opera-}\\ \text{tions, but different objects}\\ \text{and axioms}\end{matrix}\right).$$

Knot Theory Anchors.


Vassiliev

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".


Goussarov

Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1\wedge x = 1,\quad x\wedge 1 = x\wedge x = x, \qquad \text{(appetizers)}$$
$$(x\wedge y)\wedge z = (x\wedge z)\wedge(y\wedge z). \qquad \text{(main)}$$

proj Q is a graded Lie algebra: set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x}\wedge\bar{y})\wedge\bar{z} = (\bar{x}\wedge\bar{z})\wedge\bar{y} + \bar{x}\wedge(\bar{y}\wedge\bar{z}).$$

An Expansion is $Z\colon \mathcal{O} \to \text{proj}\,\mathcal{O}$ s.t. $Z(I^n) \subset (\text{proj}\,\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}} = Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho : \mathcal{A} \to \text{proj}\,\mathcal{O}$. So find $Z'\colon \mathcal{O} \to \mathcal{A}$ with $Z'(I^n) \subset \mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}} \circ \rho_n = Id_{\mathcal{A}_n}$:

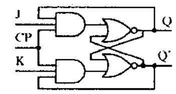
$$\begin{tikzcd}
\mathcal{O} \arrow[r, "Z'"] \arrow[dr, "Z"'] & \mathcal{A} \arrow[r, leftrightarrow] \arrow[d, "\rho"] & \mathcal{A}_n \\
& \text{proj}\,\mathcal{O} \arrow[r, leftrightarrow] & (\text{proj}\,\mathcal{O})_n \arrow[u, "Z'_{I^n/I^{n+1}}"']
\end{tikzcd}$$

Can you make this diagram less confusing?


X–S. Lin

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.

The Key Point. If $\mathcal{O}$ is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.



A J-K Flip Flop

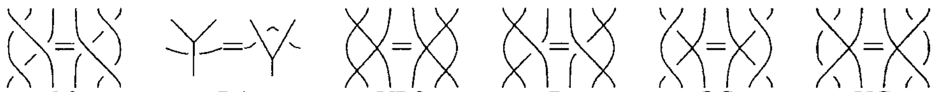
* Have "circuits" with "ends"
* Can be wired arbitrarily.
* May have "relations" – de-Morgan, etc.

further operations: delete, unzip.

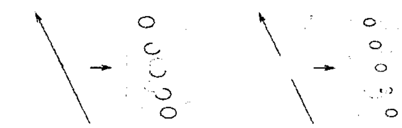
$wTT = CA\langle \ \rangle /$ R123, R4 (for vertices), F, OC.

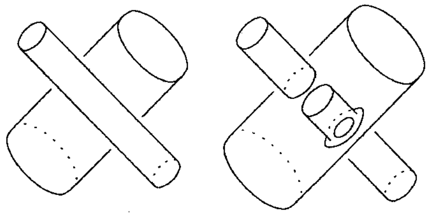
$= PA\langle \ \rangle /$ R1234, F, VR1234, D, OC.

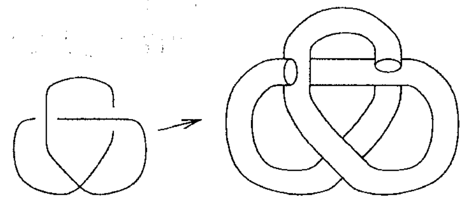
(=tangles in thick surfaces, modulo stabilization)



R3 R4 VR3 D OC UC







(and π_1 is preserved)

$(R, F) \longleftrightarrow (\ ,\ )$ $(r, t) \longleftrightarrow (\ ,\ )$

$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \longleftrightarrow$

$FF^! = I \longleftrightarrow$ $\xrightarrow{\text{unzip}}$

$F^{-1}e(x+y)F = e(x)e(y)$

$\Updownarrow$

$F^{23}R^{1,23} = R^{12}R^{13}F^{23} \longleftrightarrow$

$R^{12,3} = R^{13}R^{23}$

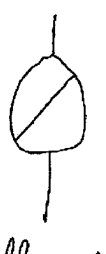
$\Updownarrow$

$F^{12,3}R^{12,3} = R^{13}R^{23}F^{12,3} \longleftrightarrow$

(unforbidding FI makes this automatic)

$RF^{21}e(-t) = F \longleftrightarrow$

$\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1}F^{2,13}F^{1,23} \longleftrightarrow$



="The tetrahedron"

Φ esdr $\longleftrightarrow$

The pentagon and the hexagons follow, with a minor twist, from the fact that we have an unzip-behaved invariant of KTG's.

For the Experienced

(and sharp-eyed)

As we did for quandles, substitute

$\mapsto$ $+ (\ - \ ) =$ $+$

into the various moves, to get relations. Also switch to "arrow diagram language": $\longleftrightarrow$. Get:

$=$ $\mapsto$ $=$ (tails commute)

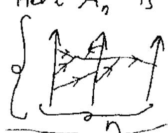
R3 $\longmapsto$ $-$ $=$ $-$ (4T really 6T)

R4 $\longmapsto$ $=$ $= 0$ (vertex invariance)

$\mathcal{A}_n^{wt}$ is $\mathcal{A}_n^{cc}$ is $\mathcal{U}(\text{tder}_n)$.

(Pretend wheels die: $= 0$)

Here $\mathcal{A}_n^{cc}$ is



/rels

trivalent directed trees with only 2-in 1-out vertices

In tensorland, this is "Co-commutative Lie-bialgs"

Rels: tails commute $=$ | Heads satisfy the only possible STU: $-$ $=$ | $-$ $=$ +also IHX and vertex invariance

$\mapsto$ $+$

α is an injection on $\mathcal{A}_n^{tree} \cong \mathcal{U}(\text{sder}_n)$. Furthermore, there is a simple charactarization of im α, so we can tell "an arrowless element" when we see it.

(approximate, false as stated) F's in Sol_0^T are in a bijective correspondance with tree-level associators for ordinary paranthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for trivalent w-tangles / with solutions of the Kashiwara-Vergne problem.

Restricted to knots, we get precisely the Alexander polynomial.

Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

Edit!

http://katlas.org

This handout and further links are at http://www.math.toronto.edu/~drorbn/Talks/MSRI-0808/