Broken Lefschetz fibrations on smooth 4-manifolds

8/12/08

Denis Auroux

Plan:

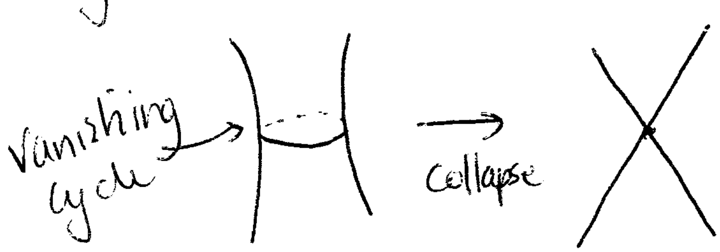
1) Introduce Broken Lefschetz fibrations (BLF)
2) Topological constructions
3) Perutz's "Lagrangian matching invariants"

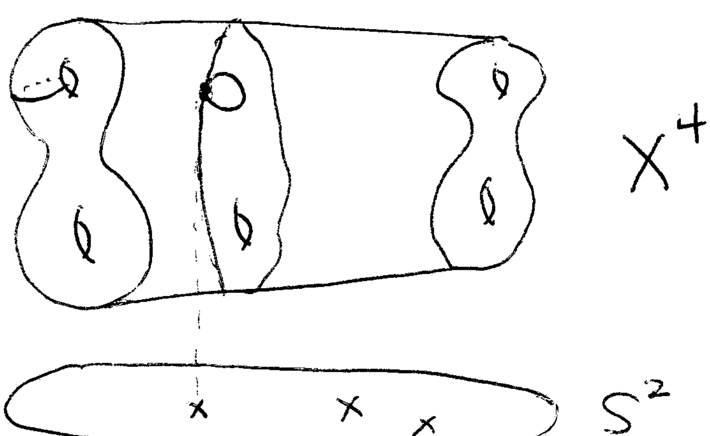
1) BLF

All mnfds are closed, oriented

Lefschetz fibration
$$f: X^4 \longrightarrow S^2$$
with all singularities modeled on
$$(z_1, z_2) \mapsto z_1^2 + z_2^2$$

Singular fibers have nodal singularies





LF $\longleftrightarrow$ positive relation among Dehn twists in mapping class gp.

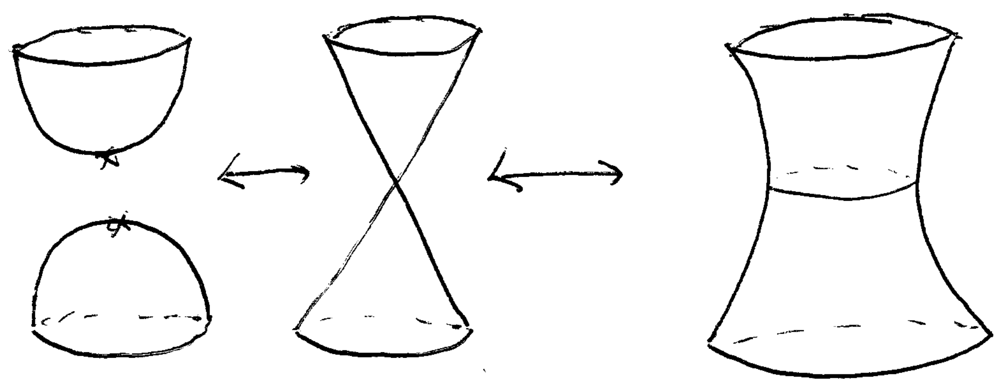
LF $\longleftrightarrow$ symplectic 4-mnfds

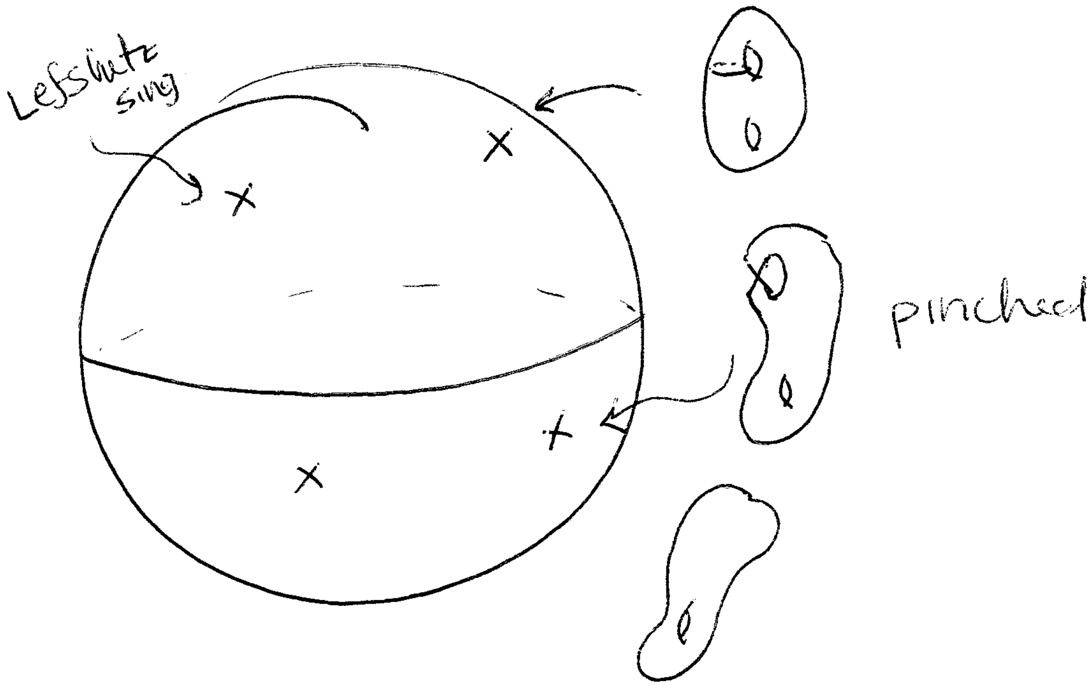
Donaldson $\longrightarrow$ (X^4, ω) symplectic $\Rightarrow$ LF on a blowup of X

Gompf $\longrightarrow$ LF, $\exists\, \alpha \in H^2(X)$, $\alpha \cdot F > 0$ $\Rightarrow$ X is symplectic (F, fiber)

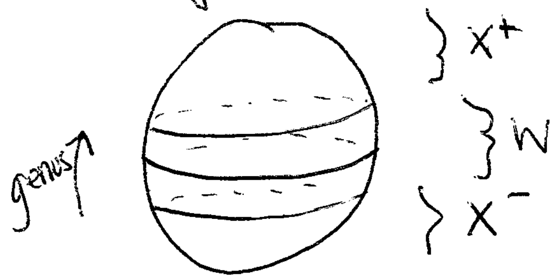
<u>Broken LF</u>: $f: X^4 \longrightarrow S^2$ with
i) Lefschetz singularies
ii) indefinite folds
"broken singularities"

$$(x, y, z, t) \longmapsto (t, x^2 + y^2 - z^2)$$





Say a BLF is <u>equatorial</u> if



then $X = X_+ \cup W \cup X_-$

$X_\pm$ genus $g_\pm$ Lefschetz fibrations over D^2, hence symplectic

$$\begin{array}{ccc} Y & \longrightarrow & W \\ & & \downarrow \\ & & S^1 \end{array}$$

fibered cobordism

$Y =$ 

<u>Near</u> <u>symplectic</u> <u>structure</u> $\omega \in \Omega^2(X)$

ω closed, $\begin{cases} \omega \text{ vanishes on } \Gamma = \amalg S^1 \\ \omega \text{ symplectic on } X - \Gamma \end{cases}$

X admits such $\omega \iff$ oriented and $b_2^+ \geq 1$ [Honda, LeBrun...]

<u>Auroux</u>-<u>Donaldson</u>-<u>Katzar</u>kov

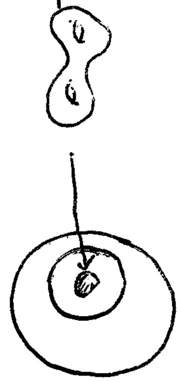
up to blowing up, near symplectic $\longleftrightarrow$ BLFs with essential fibers

- (X, ω, Γ) near symplectic $\Rightarrow$ a blow-up of X admits an equitorial BLF, with broken singularities equal to Γ.
- Given a BLF, $\exists\ \alpha \in H^2(X) / \alpha \cdot$[comp. of fibers] > 0

then X carries a near symplectic ω with $f^{-1}(0) \equiv$ broken singularities.

2) <u>Topological</u> <u>constructions</u>

Gay-Kirby - build BLFs from handle decomp.



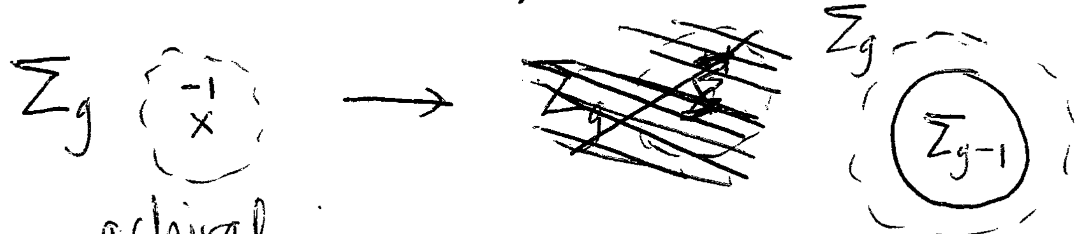
$\Sigma \times D^2 \longrightarrow$ "grow" by

- lefschetz sing.: attach 2-handle with framing -1
- circle sing.: attach round 1-handle, 2-handle

Thm (Gay-Kirby)
For all X^4 oriented, closed, $\forall F \subseteq X$ a surface with $F^2 = 0$, ~~there~~ then exists achiral BLF $f: X \to S^2$ with F, a fiber.

achiral Lefschetz = mirror image of Lefschetz

Observation: (Perutz)



achiral Lefschetz sing.

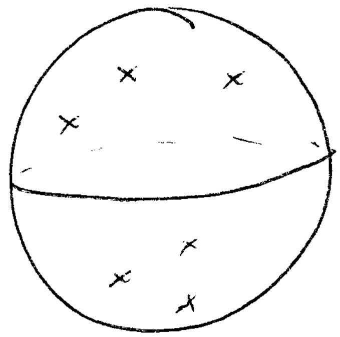
broken lefschetz sing.

Corollary: a blowup of X carries a BLF.

Y. Lekili: study deformations of BLFs by homotopies

approach: singularity theory

3) Perutz's Lagrangian matching invariants



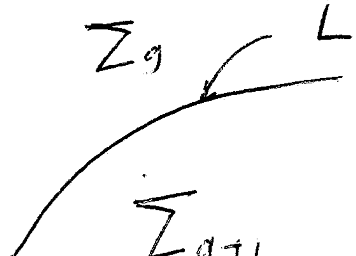
Σ_g near symplectic

Σ_{g+1} count pseudoholomorphic multisections of f with boundary Γ

$$SW_X(s) \overset{?}{=} \mathcal{L}_{X,f}(s) \in \mathbb{Z} \qquad s \in \mathrm{spin}^c(X) \text{ admissible}$$

over U, $\mathrm{Sym}^r(f)$

over V, $\mathrm{Sym}^{r+1}(f)$ $\qquad r = \frac{1}{2}\langle c_1(s), \text{fiber}\rangle + g - 1$



Σ_g $\qquad L \subseteq \mathrm{Sym}^r \Sigma_g \times \mathrm{Sym}^{r+1} \Sigma_{g+1}$ Lagrangian submanifold

Σ_{g+1}

count sections of $\mathrm{Sym}^r f \to U$

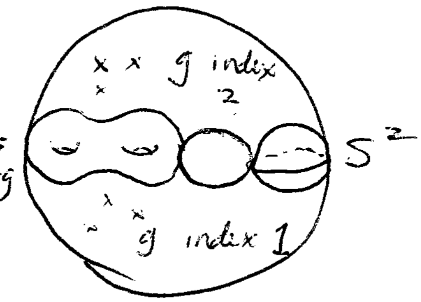
$\mathrm{Sym}^{r+1} f \to V$

which are in L over the equator

Recent work in progress (Y. Lekili)

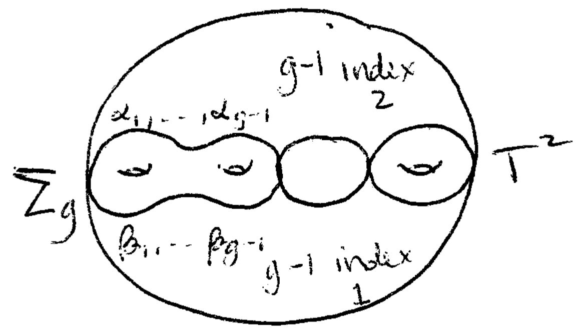
- try to relate Perutz TQFT to Heegaard Floer ~~homology~~ theory

Perutz observed $\begin{array}{c} M^3 \\ f \downarrow \\ S^1 \end{array}$



$$HF(f) \cong HF(T_\alpha, T_\beta) \text{ in } \mathrm{Sym}^g \Sigma_g$$

$\begin{array}{c} M^3 \\ \downarrow \\ S^1 \end{array}$



$$T_\alpha = \alpha_1 \times \cdots \times \alpha_{g-1} \subseteq \mathrm{Sym}^{g-1} \Sigma_g$$
$$T_\beta = \beta_1 \times \cdots \times \beta_{g-1}$$

Conjecture:
Restricting to spin^c structures with $\langle c_1(s), \text{fiber} \rangle = 0$
Novikov "$HF(f) \cong HF(T_\alpha, T_\beta)$ in $\mathrm{Sym}^{g-1}\Sigma_g$"

$$HF^{Nov}(T_\alpha, T_\beta) \cong HF^+(M, \eta) \text{ in } \mathrm{Sym}^{g+2}(\Sigma_{g+2})$$

in $\mathrm{Sym}^{g-1}\Sigma_g$ perturbed HF^+

$[\eta] \cdot \text{fiber} > 0$

Broken Lefschetz fibrations on smooth 4-mflds – MSRI, August 12, 2008

[thank ORGANIZERS ; Rob for motivating me to keep looking at this]

Goal. Survey of various results around BLF's

Plan: 1) broken Lefschetz fibrations
2) topological constructions
3) Perutz's "Lagr. matching invariants"

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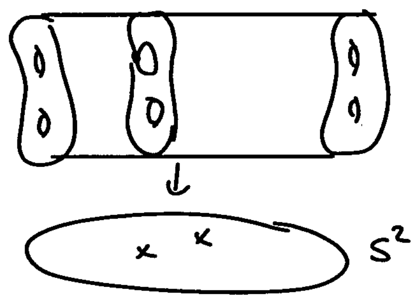
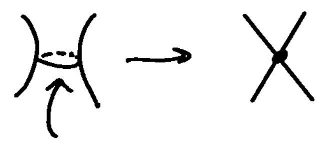
1. BLF's [all mflds closed, oriented]

- Lefschetz fibrations : $f: X^4 \longrightarrow S^2$ compact oriented
with all singularities modelled on $(z_1, z_2) \mapsto z_1^2 + z_2^2$
$\mathbb{C}^2 \longrightarrow \mathbb{C}$



☆ sing. fibers have nodal sings.



vanishing cycle collapse to pt
monodromy = +ve Dehn twist.

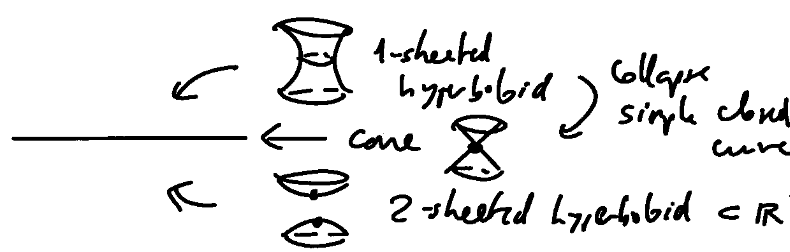
Donaldson + Gompf: symplectic mflds $\Longleftrightarrow$ LF's
- Donaldson: (X, ω) sympl $\Rightarrow$ LF on a blowup of X
- Gompf: LF, $\exists\, \alpha \in H^2(X) \,/\, \alpha \cdot$ [components of fibers] $> 0 \Rightarrow$
$\exists$ sympl. form on X, canonical up to deform^n, st. $\omega_{|\text{fiber}} > 0$.

Algebraically, genus g LF $\Longleftrightarrow$ positive relation among Dehn twists
$\tau_1 \cdots \tau_r = Id$. in MCG

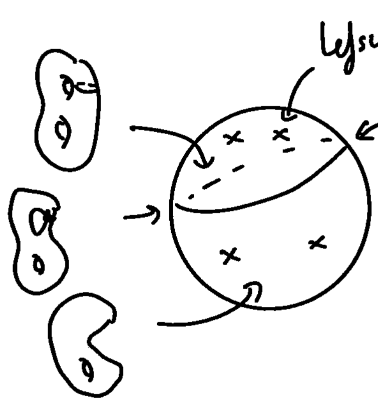
- Broken LF : $f: X^4 \to S^2$ with $\begin{cases} \cdot \text{ Lefschetz singularities} \\ \cdot \text{ indefinite fold ("broken") singularities} \end{cases}$

local model:

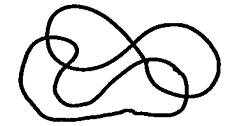
$(x, y, z, t) \longmapsto (t, x^2 + y^2 - z^2)$



globally:



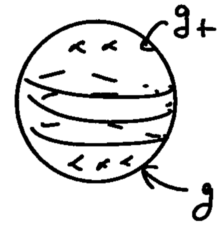
NB: in general BLFs, $f|_{\mathrm{sing}\,f}$ only immersion, so in S^2 can have intersecting circles



nicer version := all circles are embedded & parallel to equator

"equatorial BLF" genus ↗ continuously from south to North

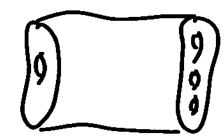
all Lefschetz sings. are in polar regions



Then get $X = X_- \cup W \cup X_+$

$X_\pm$ = genus $g_\pm$ Lefschetz fibrations / D^2 (symplectic)

$Y \to W$, $W \downarrow S^1$ fibered cobordism, $Y =$



Near-symplectic structure := ω closed 2-form

$\omega \begin{cases} \text{vanishes on } \Gamma = \sqcup S^1\text{'s} \\ \text{symplectic on } X \setminus \Gamma \end{cases}$

(standard model near Γ).

- X admits such ω $\iff$ orientable & $b_2^+ \geq 1$. [Honda, LeBrun, ...]

Auroux-Donaldson-Katzarkov: up to blowing up, near-sympl. mflds $\leftrightarrow$ BLFs

HARD, NOT EXPLICIT (ANALYSIS)

- (X, ω, Γ) near symplectic $\Rightarrow$ a blowup of X admits an equatorial BLF with circle sings $= \Gamma$.

- BLF, $\exists \alpha \in H^2(X)$ s.t. $\alpha \cdot [\text{components of fibers}] > 0 \Rightarrow X$ carries a near sympl. ω, canonical up to deform$^{\underline{n}}$, $\omega|_{\text{fibers}} > 0$, $\omega^{-1}(0) \equiv$ broken sings.

EASY

2. Topological constructions

- Gay Kirby set out to build BLFs by topology – using handlebody decomp^s.
  - NOTE: starting from $\Sigma \times D^2$ and slowly "growing" the fibration to get preimage of larger subsets of S^2

    Lefschetz sing $\longleftrightarrow$ 2-handle with framing -1

    broken sing $\longleftrightarrow$ round 1-handle ($=$ 1-handle $\times S^1$)

  - boundary of a BLF over D^2 w/ open fibers ("convex BLF") is naturally an open book! (pages $=$ fibers above ∂D^2, binding $= \partial(\text{fiber})$)

    $\rightsquigarrow$ try to build two halves of BLF then match boundaries...

    use Eliashberg & Giroux to get the open books to match after stabilizations.

Doesn't quite work all the way:

Thm (Gay-Kirby) ‖ $\forall X^4$ oriented closed 4-mfld, $\forall F \subset X$ surface w/ $F^2 = 0$, $\exists$ achiral BLF $f: X \to S^2$ with a fiber $= F$

ie. allow achiral Lefschetz sings $=$ mirror image of Lefschetz [very bad for symplectic purposes...]

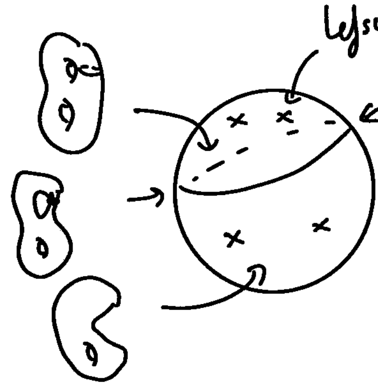
But... observation (Perutz) $\quad$ $\times^{-1}$ achiral Lefschetz v.c. $\gamma \subset \Sigma_g$ on X $\longrightarrow$ Σ_g Σ_{g-1} broken, same v.c. $\gamma \subset \Sigma_g$ on $X \# \overline{\mathbb{CP}^2}$

Corollary: ‖ after blowups, get a BLF

Since then, a lot over the last few months!!

globally:

Lefschetz singularities



broken singularities – <u>genus of fiber jumps</u> by 1

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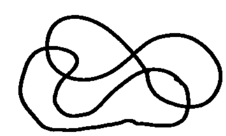

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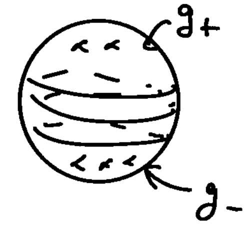

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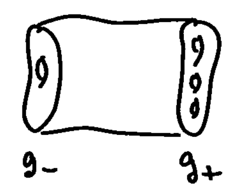
$\begin{matrix} Y \to W \\ \downarrow \\ S^1 \end{matrix}$ fibered cobordism, $Y =$



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<u>4 Donaldson-Katzarkov:</u> up to blowing up, near-sympl. mflds $\leftrightarrow$ BLFs

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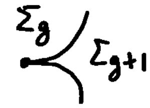
- <u>Y. Lekili:</u> study <u>deformations</u> of BLFs by homotopies

  Approach: sing. theory. <u>BUT:</u> Lefschetz sing. are not generic!
  generic maps have folds & cusps!

  → as a device, consider "<u>wrinkled fibrations</u>" := BLF + allow indefinite cusps.

  $(x,y,z,t) \longmapsto (t,\ x^3 - xt + y^2 - z^2)$



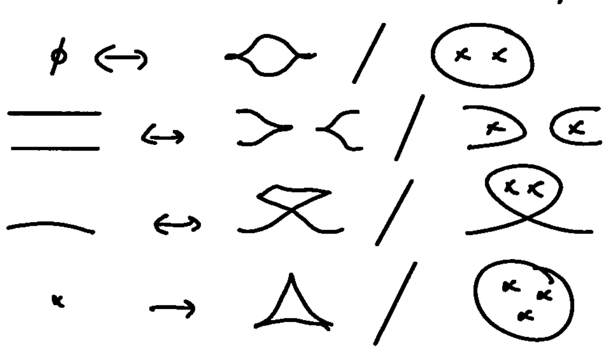
★ <u>Prop</u> (Lekili) ‖ ∃ homotopic deformation

⟷

cusp — broken Lefschetz.

★ set of moves on wrinkled fibrations (from classical sing theory) + convert them to moves (homotopies) on BLFs

<u>4 moves:</u>



<u>Conj:</u> they're enough to connect any two homotopic BLFs

- <u>I. Baykur:</u> ‖ Any smooth 4-mfld carries a BLF

  <u>Warning</u>: set of sing. circles ⊂ S^2 very complicated, not at all equatorial!!!

  Proof is simple, combines 2 ingredients

  1) <u>Saeki</u>: elimination of definite singularities:
     any map to S^2 is homotopic to one with only indefinite fold & cusp singularities – ie, in Lekili's terminology, a wrinkled fibration

  2) <u>Lekili</u>: homotope wrinkled ⇝ BLF as above.

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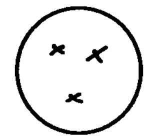
- shortly after, Lekili:

Prop: ∃ homotopic deformation × → 
achiral Lefschetz — broken + 3 Lefschetz.

start with mirror image of × (Lefschetz) → 
3 cusps

but cusp model has orient.-reversing self-diffeo: so we get inward cusps!! Then  ×3.

Corollary: (Gay-Kirby + Lekili)

Every X^4 admits an equatorial BLF.

- independently, Akbulut-Karakurt: same result by direct argument using handlebody decomp^ns & open books — in a sense, this is the proof that Gay-Kirby were aiming for but didn't quite get.
- Jonathan Williams is working on making BLFs equatorial → might improve Baykur's approach to get equatoriality....

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3. Perutz Lagr. matching invariants

generalizes Donaldson-Smith invariants of Lefschetz fibrations from symplectic (LF) to near-symplectic (BLF) setting.

DS counts pseudoholom. sections of $Hilb^r(f)$ = higher-dim^l sympl. fibration with smooth fiber = Sym^r(fiber of f).

Usher: DS = Gromov-Taubes, hence indep^t of LF and = SW

- Y. Lekili: study deformations of BLFs by homotopies

  Approach: sing. theory. But: Lefschetz sing. are not generic!
  generic maps have folds & cusps!

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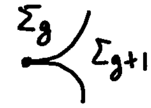

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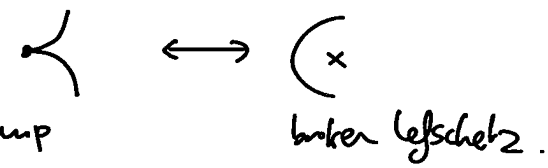
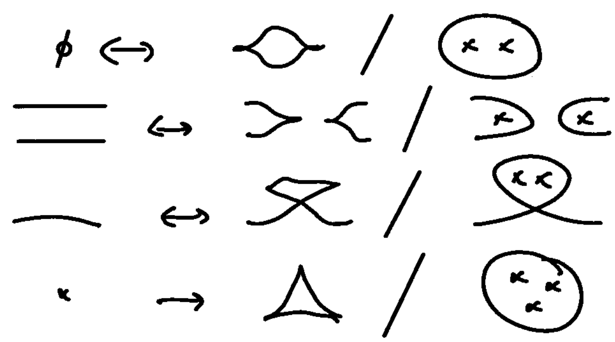


  ★ set of moves on wrinkled fibrations (from classical sing theory)
  + convert them to moves (homotopies) on BLFs

  4 moves:



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  1) Saeki: elimination of definite singularities:
     any map to S^2 is homotopic to one with only indefinite fold & cusp singularities – ie, in Lekili's terminology, a wrinkled fibration

  2) Lekili: homotope wrinkled ⇝ BLF as above.

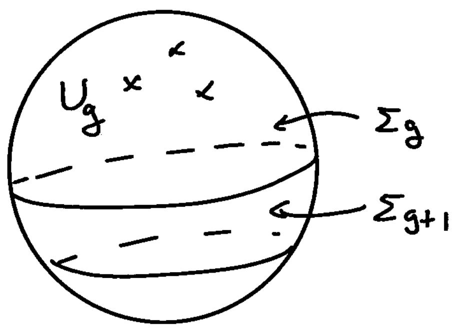
<u>Perutz:</u> BLF $\quad f : X \to S^2$ w/ sing. circles Γ

$\Rightarrow$ count pseudoholomorphic "multisections" of f with boundary on Γ

(in line w/ Taubes' program, these should $\leftrightarrow$ SW solutions)

$\mathcal{L}_{X,f}(s) \in \mathbb{Z}$ for $s \in spin^c(X)$ satisfying admissibility condition

$\triangle$ not known to be indep. of BLF f !!

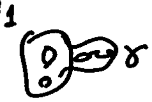


- Over $U_g \subset S^2$ where fibers of f are Σ_g, consider $\mathrm{Hilb}^r(f)$ where $r = \frac{1}{2}\langle c_1(s), [\text{fiber}]\rangle + g - 1$

- over a separating circle

$$\Sigma_g \xleftarrow{\text{collapse loop } \gamma \subset \Sigma_{g+1}} \Sigma_{g+1}$$

$$\mathrm{Hilb}^r \qquad \mathrm{Hilb}^{r+1}$$



can define a Lagr. submfld $V \subset \mathrm{Sym}^r \Sigma_g \times \mathrm{Sym}^{r+1} \Sigma_{g+1}$

(<u>roughly</u>: r pts in $\Sigma_{g+1} \setminus \gamma \simeq \Sigma_g - 2$ pts
1 pt in γ
... but not quite !).

Then $\mathcal{L}_{X,f}(s)$ counts pseudo holom. sections of $\mathrm{Hilb}^r(f)$ over each region U_g delimited by critical set, s.t. along common boundary they satisfy <u>matching condn</u>: $\forall p \in \partial U_g \cap \partial U_{g+1}$,

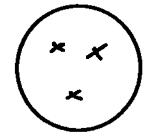
$U_g \;\Big|\; U_{g+1}$ $\qquad (\sigma_g(p), \sigma_{g+1}(p)) \in V_p \subset \mathrm{Sym}^r \Sigma_g \times \mathrm{Sym}^{r+1} \Sigma_{g+1}$

* Various properties similar to SW invariants
* Fit in a TQFT wrt cutting 4-mflds + BLF along 3-mflds with "broken fibration" over S^1, ie. S^1-valued Morse fn without definite critical pts

$f: M^3 \to S^1 \rightsquigarrow HF(f, s)$ $\qquad$ (• for mapping torus of ϕ, it's $= HF(\mathrm{Sym}^r \phi)$)

and BLF over surface w/ boundary $\rightsquigarrow$ relative invariants.

• shortly after, Lekili:

Prop: ∃ homotopic deformation × → 
achiral Lefschetz → broken + 3 Lefschetz.

start with mirror image of × → 
Lefschetz → 3 cusps

but cusp model has orient.-reversing self-diffeo: so we get wrinkled cusps!! Then  ×3.

Corollary: (Gay-Kirby + Lekili)

Every X^4 admits an equatorial BLF.

• independently, Akbulut-Karakurt: same result by direct argument using handlebody decomp^{ns} & open books — in a sense, this is the proof that Gay-Kirby were aiming for but didn't quite get.

• Jonathan Williams is working on making BLFs equatorial
→ might improve Baykur's approach to get equatoriality ....

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3. Perutz Lagr. matching invariants

generalizes Donaldson-Smith invariants of Lefschetz fibrations from symplectic (LF) to near-symplectic (BLF) setting.

DS counts pseudoholom. sections of $\mathrm{Hilb}^r(f)$ = higher-dim^{l} sympl. fibration with smooth fiber = Sym^r(fiber of f).

Usher: DS = Gromov-Taubes, hence indep^t of LF and = SW

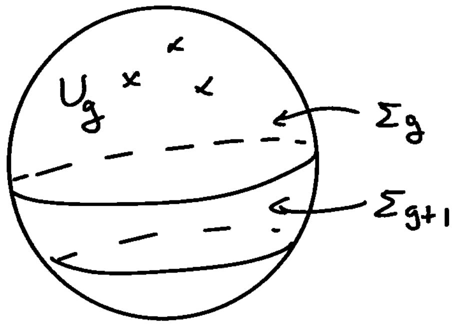
<u>Perutz:</u> BLF $f : X \to S^2$ w/ sing. circles Γ

$\Rightarrow$ count pseudoholomorphic "multisections" of f with boundary on Γ

(in line w/ Taubes' program, these should $\longleftrightarrow$ SW solutions)

$\mathcal{L}_{X,f}(\mathfrak{s}) \in \mathbb{Z}$ for $\mathfrak{s} \in \mathrm{spin}^c(X)$ satisfying admissibility condition

⚠ not known to be indept. of BLF f !!



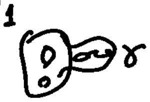
- Over $U_g \subset S^2$ where fibers of f are Σ_g, consider $\mathrm{Hilb}^r(f)$ where $r = \frac{1}{2}\langle c_1(\mathfrak{s}), [\text{fiber}]\rangle + g - 1$

- over a separating circle

collapse loop $\gamma \subset \Sigma_{g+1}$

$\Sigma_g \leftarrow \Sigma_{g+1}$

$\mathrm{Hilb}^r \qquad \mathrm{Hilb}^{r+1}$



can define a Lagr. submfld $V \subset \mathrm{Sym}^r \Sigma_g \times \mathrm{Sym}^{r+1} \Sigma_{g+1}$

(<u>roughly</u>: r pts in $\Sigma_{g+1} \setminus \gamma \simeq \Sigma_g - 2$ pts
1 pt in γ
... but not quite !)

Then $\mathcal{L}_{X,f}(\mathfrak{s})$ counts pseudoholom. sections of $\mathrm{Hilb}^r(f)$ over each region U_g delimited by critical set, s.t. along common boundary they satisfy <u>matching condⁿ</u>: $\forall p \in \partial U_g \cap \partial U_{g+1}$,

$U_g \mid U_{g+1}$ $\qquad (\sigma_g(p), \sigma_{g+1}(p)) \in V_p \subset \mathrm{Sym}^r \Sigma_g \times \mathrm{Sym}^{r+1} \Sigma_{g+1}$

* Various properties similar to SW invariants
* Fit in a TQFT wrt cutting 4-mflds + BLF along 3-mflds with "broken fibration" over S^1, ie. S^1-valued Morse fⁿ without definite critical pts.

$$\begin{array}{c} M^3 \\ f \downarrow \\ S^1 \end{array} \rightsquigarrow HF(f, \mathfrak{s}) \qquad (\bullet \text{ for mapping torus of } \phi, \text{ it's } = HF(\mathrm{Sym}^r \phi))$$

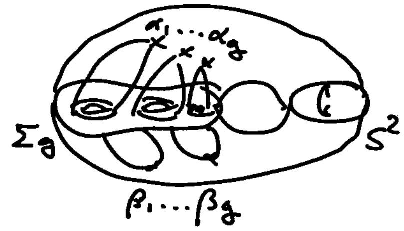
and BLF over surface w/ boundary $\rightsquigarrow$ relative invariants.

* Recent work in progress (Y. Lekili):

try to relate Perutz TQFT to Heegaard-Floer theory

- Perutz observed: $\begin{matrix} M^3 \\ \downarrow f \\ S^1 \end{matrix}$ with g index 1 crit pts, g index 2

lowest genus fiber is S^2



$$\rightsquigarrow HF(f) \simeq HF(T_\alpha, T_\beta) \text{ in } \mathrm{Sym}^g \Sigma_g$$

- Next case: $\begin{matrix} M^3 \\ \downarrow f \\ S^1 \end{matrix}$ with lowest genus fiber $= T^2$

$\left.\begin{matrix} g-1 \text{ index } 1 \text{ crit. pts} \\ g-1 \text{ index } 2 \end{matrix}\right\} \rightarrow \begin{matrix} \alpha_1 \ldots \alpha_{g-1} \subset \Sigma_g \\ \beta_1 \ldots \beta_{g-1} \end{matrix}$

$$T_\alpha = \alpha_1 \times \cdots \times \alpha_{g-1} \subset \mathrm{Sym}^{g-1} \Sigma_g$$
$$T_\beta = \beta_1 \times \cdots \times \beta_{g-1} \subset$$

Restricting to spin^c structures with $\langle c_1(s), [\text{fiber}] \rangle = 0$,

$HF(f) \cong HF(T_\alpha, T_\beta)$ in $\mathrm{Sym}^{g-1}(\Sigma_g)$ (⚠ monotonicity issue - use suitable Novikov ring)

<u>Conj:</u> Restricting to $\langle c_1(s), [\text{fiber}] \rangle = 0$,

$HF(T_\alpha, T_\beta) \simeq HF^+(M, \eta)$ perturbed HF^+ by $[\eta].[\text{fiber}] > 0$

in $\mathrm{Sym}^{g-1} \Sigma_g$ in $\mathrm{Sym}^{g+2}(\Sigma_{g+2})$ (both over $\mathbb{Z}[t^{-1}, t]]$)

(in progress - most ingredients of proof now in place)

Also expect elementary cobordism maps to match up nicely; would get:

<u>Conj:</u> $f : X \to S^2$ equatorial BLF with lowest genus fiber T^2

$$\Rightarrow \mathcal{L}_{(X,f)} = OS_X$$

(⚠ again, need perturbations to deal with lack of monotonicity)