8/13/08

# Unified Witten-Reshetikhin-Turaev invariants of rational homology 3-spheres

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**Plan:**

1. Definition of WRT invariants
2. Unification
3. Proofs

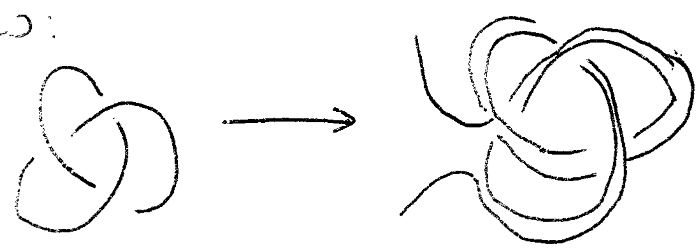
## ① WRT

ingredients: © Kirby calculus

$$\{\text{framed links}\} \Big/ K1, K2 \xrightarrow{1-1} \{\text{closed oriented 3-mnfds}\}$$

relations:

K1



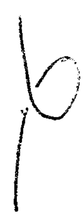
K2 $\quad L \longrightarrow L \sqcup \Big\{ \begin{matrix} \infty \\ \infty \end{matrix}$

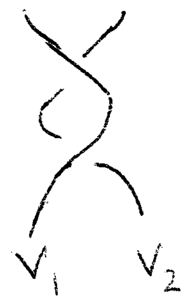
b. Colored Jones polynomial

$R \in \mathcal{U}_q(sl_2)$ $\quad R : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$

$\#$

$R^{-1} : V_2 \otimes V_1 \longrightarrow V_1 \otimes V_2$

 $= q^{\frac{n^2-1}{4}}$ 



$=$



$J_L(n_1, \dots n_m) \in \mathbb{Z}[q^{\pm 1}]$

So WRT

$$\sum_{n_i=1}^{2r} \prod_{i=1}^{m} [n_i]\, J_L(n_1, \dots, n_m) =: F_L(\xi)$$

invariant under K1 move

where $[n] = \dfrac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ $\qquad q = \xi, \ \xi^r = 1$

$M = S^3(L)$

$$\tau_M(\xi) = \frac{F_L(\xi)}{\left(F_{U_+}(\xi)\right)^{\sigma_+} \left(F_{U_-}(\xi)\right)^{\sigma_-}} \in \mathbb{C}$$

$\sigma_{\pm} = \#$ positive / negative e'values of $L$

- Not a compute set of invariants
- $\tau_M(\xi) \in \mathbb{Z}[\xi]$ ord $\xi = p$, prime
- Ohtsumi series $M$ is QHS and ord $\xi = p > |H_1(M)| = 6$

$$\tau_M(\xi) = c_{r,0} + c_{r,1}(1-\xi) + c_{r,2}(1-\xi)^2 + \dots + c_{r,r-2}(1-\xi)^{r-2}$$

$$f = \sum_{i=0}^{\infty} \lambda_i (1-q)^i \in \mathbb{Q}[[1-q]]$$

$$\begin{tikzcd}
 & \mathbb{Z}[q^{\pm 1}] \arrow[dr, "q="] & \\
\{\text{lines}\} \arrow[ur] \arrow[dr] & & \mathbb{Q}[[h]] \\
 & a \arrow[ur] \arrow[d, "\text{surgery}"] & \\
 & \mathbb{C} & \\
\{\text{3-mfld}\} \arrow[ur] \arrow[dr] & & \mathbb{Q}[[h]] \\
 & a \arrow[ur] &
\end{tikzcd}$$

Thm.
Let $M$ be a $\mathbb{Q}HS$, $|H_1(M,\mathbb{Z})| = b$,
then there exists $I_M \in \mathcal{R}_b$ s.t.
$\omega_\xi(I_M) = \tau_M(\xi)$.

Ring $\mathcal{R}_b$

① $b = 1$

$$I_{n+1} \subseteq I_n = \big( (1-q) \cdots (1-q^n) \big) \in \mathbb{Z}[q]$$

$$R_1 = \widehat{\mathbb{Z}[q]} = \lim_{\overleftarrow{n}} \frac{\mathbb{Z}[q]}{I_n}$$

$$f = \sum_{k=0}^{\infty} f_k(q)(1-q) \cdots (1-q^n)$$

$\omega_\xi f$ is well-defined

$$(q^a)_b = (1-q^a)(1-q^{a+1}) \cdots$$

$$(q)_n' \subseteq I_{\lfloor \frac{n}{2} \rfloor} \qquad f' \in \widehat{\mathbb{Z}[q]}$$

Thm:
$$T_\xi : \widehat{\mathbb{Z}[q]} \hookrightarrow \mathbb{Z}[\xi][[q-\xi]] \quad \text{injection}$$
$$f \longmapsto \sum_{n=0}^{\infty} \frac{f^{(n)}(\xi)}{n!} (q-\xi)^n$$

$\omega_\xi f =$ ord $\xi = p, p^2, p^3, \ldots$ determines $f$.

② $b = \prod_{i=1}^{s} p_i^{k_i}$, $S \subseteq \mathbb{N}$

$\Phi_S = \{ \Phi_n(q) \mid n \in S \}$

$\Phi_S^*$ mult set generated by $\Phi_S$ ordered by divisibility

$\mathbb{Z}[q]^S = \varprojlim \frac{\mathbb{Z}[q]}{f(q)}$ $\quad N_b = \{ n \in \mathbb{N} \mid (n,b)=1$

Ex:

$\mathbb{Z}[q]^{\mathbb{N}} = \widehat{\mathbb{Z}[q]}$

$R_b = \mathbb{Z}[\frac{1}{b}][q]^{\mathbb{N}_{odd}} = \prod_{K} \mathbb{Z}[\frac{1}{b}][q]^{p_1^{k_1} \cdots p_i^{k_i} \mathbb{N}_b}$

If $b$ is odd, $\mathbb{Z}[\frac{1}{2b}]$

Thm: Beliakova, Le

$\mathbb{Z}[\frac{1}{b}][q]^{p^k N_b} \hookrightarrow \mathbb{Z}[\frac{1}{b}][[1-q^{p^k}]]$

$$
\begin{tikzcd}
 & \mathbb{C} & \\
\left\{ \begin{array}{c} M \in \mathbb{Q}HS \\ |H_1(M)| = b \end{array} \right\} \arrow[ur, "\sigma"'] \arrow[r, "I_M"] \arrow[dr] & R_b \arrow[u, "\omega_\xi"'] \arrow[r, hook] & \prod_{c \mid b} \mathbb{Z}[\tfrac{1}{b}][[1-q^c]] \arrow[d, leftrightarrow] \\
 & \mathcal{A} \arrow[r] & \mathbb{Q}[[h]]
\end{tikzcd}
$$

<u>Corollary</u>. $\tau_M(\xi) \in \mathbb{Z}[\xi] \quad \forall \xi, \forall M$

For $M$ a $\mathbb{Q}HS$, $|H_1(M)| = b$

1. $LMO_M$ determines $\{\tau_M(\xi) \mid (\text{ord } \xi, b) = 1\}$

2. $\exists$ generalized Ohtsuki series $\mathbb{Z}[\frac{1}{2b}][[1-q^c]]$ dominating WRT $\{\tau_M(\xi) \mid (\text{ord } \xi, b) = c\}$

3. $\{\tau_M(\xi), \forall \xi\}$ is determined

4. WRT separate Seifert fibered $\mathbb{Z}HS$.

## Proofs

$M = S^3(K_6)$

$$F_K(\xi) = \sum_{n=1}^{2r} [n]\, J_K(n)$$ ~~cyclotomic expansion~~

cyclotomic expression

$$J_{K_0}'(n) = \sum_{k=0}^{\infty} C_{K,k}(q)\,(q^{1-n})_k\,(q^{1+n})_k$$

$$F_K(\xi) = \sum_{k=0}^{\infty} C_{K,k}(q) \sum_{n=0}^{2r} q^{6\frac{n^2-1}{4}} (q^n)(q^{-n})$$

Lemma: (Beliakova - Le): If $(r,b)=c$

$$\sum_{n=1}^{2r} q^{6\frac{n^2-1}{4}}\, q^{a\cdot n} = \begin{cases} 0, & c \nmid a \\ q^{-\frac{a^2}{6}} \sum_{n=1}^{2r} q^{6\frac{n^2-1}{4}} \end{cases}$$

Laplace transform

$$\tilde{\mathcal{L}} \quad \mathbb{Z}[q^{\pm 1}, q^{\pm n}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{6}}$$

$$q^{an} \longmapsto \begin{cases} 0, & c \nmid a \\ q^{-\frac{a^2}{6}}, & \text{else} \end{cases}$$

$$F_K(\tau) = \sum_{k=0}^{\infty} C_{K,k}(q)\, \alpha_{b,c,n}\left( (q^{n})_{k+1} (q^{-n})_{k+1} \right)$$

Rogers - Ramanujan Identity

<u>Example:</u> $M$, Poincare sphere

$$I_M = \frac{q}{1-q} \sum_{k=0}^{\infty} (-1)^k q^{-\frac{(k+2)(3k+1)}{2}} \left(q^{k+1}\right)_{k+1}$$

$$I_{S^3(4,1,1)} = \frac{q}{1-q} \sum_{k=0}^{\infty} (-1)^k q^{-(k+1)^2} \left(q^{k+1}\right)_{k+1}$$

# Unified WRT invariants of rational homology spheres

First of all I would like to thank the organizers for the invitation. It an honor for me to be here.

After heavilly using Kirby calculus to finally ... personally

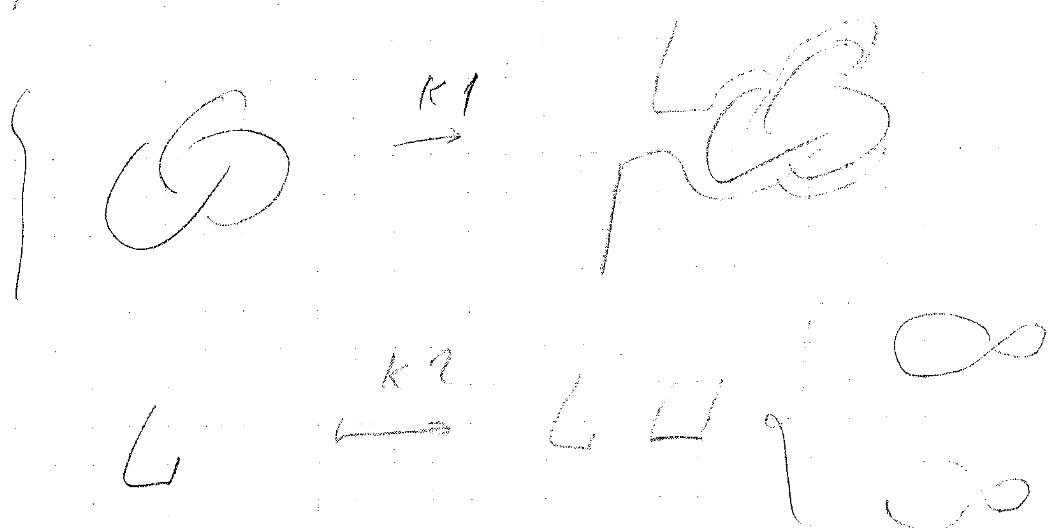
Plan

1) Definition of WRT invariants
2) Unification
3) ~~Results~~ Proofs

## ① WRT invariants

- Kirby calculus

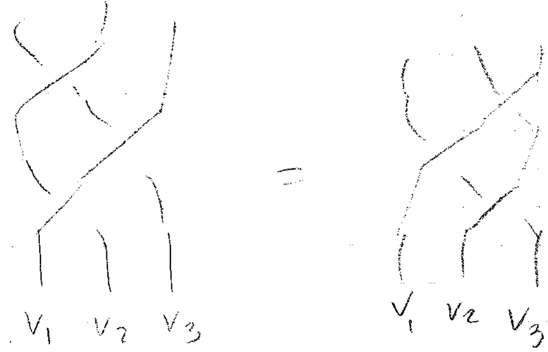
$$\left\{ \begin{matrix} \text{framed} \\ \text{links} \end{matrix} \right\} \Big/ K1, K2 \xrightarrow[\text{surgery}]{1:1} \left\{ \begin{matrix} \text{closed, oriented} \\ \text{3 mnf} \end{matrix} \right\}$$

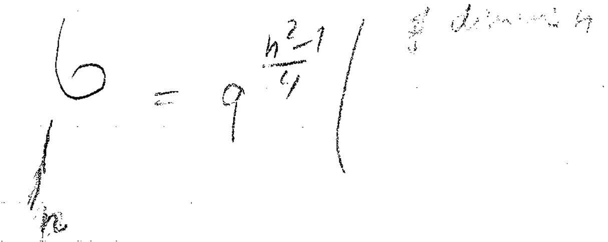



- colored Jones polynomial

$R \in U_q(sl_2)$

since for any ...





$$J_L(V_1, \dots V_m) := J_L(n_1, \dots, n_m) \in \mathbb{Z}[q^{\pm 1}]$$

? How to make it invariant under Kirby moves?

RT — take a 'state sum' over all colour

$q = \xi \qquad \xi^r = 1 \qquad r$ odd

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

$$\sum_{\substack{n_i = 1 \\ \text{odd}}}^{2r} \prod_{i=1}^{m} [n_i] \, J_L(n_1 \dots n_m) = F_L(\xi) \quad \text{invariant under KL}$$

an infinite family of invariants: one for any root of unity

$$\mathbb{C} \ni \tau_M(\xi) := \frac{F_L(\xi)}{(F_{U_+}(\xi))^{\sigma_+} (F_{U_-}(\xi))^{\sigma_-}}$$

$\sigma_\pm$ # posit. / negat. eigenvalues of linking matrix $L$

What do we know about this invariants?

- [Johanna Kania - B] they are not complete

• [Ohtsuki] $\operatorname{ord}\xi = p$, $M \in \mathbb{Q}HS$ with $|H_1(M;\mathbb{Z})| = b$

$$\tau_M(\xi) = c_{r,0} + \underset{\in \mathbb{Z} a_i =}{c_{r,1}}(1-\xi) + c_{r,2}(1-\xi)^2 + \dots c_{r,n-2}(1-\xi)$$

$\sim \frac{a}{b}$

$\exists\ \lambda_i \in \mathbb{Q}$ $\quad c_{r,i} = \lambda_i \mod r$ $\quad b^* b = 1 \mod r$

$p > b$

$= \sum \lambda_i (1-1)^i$

$\exists f \in \mathbb{Q}[[1-q]]$ Ohtsuki series

How about relation of $f$ with other invariants

Recall

$$\begin{tikzcd}
 & \mathbb{Z}[q^{\pm 1}] \arrow[dr, "q = e^h"] \arrow[loop below, "\text{Le, Murakami, Kau}"] & \\
\{\text{links}\} \arrow[ur, "J_L"] \arrow[dr, "\text{Kontsevich integral}"'] & & \mathbb{Q}[[h]] \\
 & \mathcal{A} \arrow[ur, "sl_2", "\text{weight system}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{C} & \\
\{\cancel{3\text{-mfd}}\ \mathbb{Q}HS\} \arrow[ur, "\tau_M(\xi)"] \arrow[r, "I_M"] \arrow[dr, "\langle M \rangle"'] & \mathcal{R}_b \arrow[u] & \mathbb{Z}[[1-q^b]] \arrow[l] \arrow[d] \\
 & \mathcal{A} \arrow[r] & \mathbb{Q}[[h]]
\end{tikzcd}$$

$b=1$

(Habiro, Le, B, Le-B-Bühler-B)

<u>Thm:</u> Let $M \in \mathbb{Q}HS$, $|H_1(M)| = b$

$\exists\ I_M \in \mathcal{R}_b$ s.t. $ev_\xi(I_M) = \tau_M(\xi)$

- Ring

1) $\ell = 1$ Habiro

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \frac{\mathbb{Z}[q]}{I_n}$$

analytic functions at roots of unity

$$I_{n+1} \subset I_n = ((1-q)\cdots(1-q^n)) = ((q)_n) \quad I_{n-1}$$

$$f = \sum_{k=0}^{\infty} f_k(q)\,(q)_k$$

- $ev_\xi(f)$ is well-defined

- $T_\xi : \widehat{\mathbb{Z}[q]} \longrightarrow \mathbb{Z}[\xi][[q-\xi]]$

$$[q]_n' \in I_{[\frac{n}{2}]} \Rightarrow f' \in \widehat{\mathbb{Z}[q]}$$

$$f \longmapsto \sum_{n=0}^{\infty} \frac{f^{(n)}(\xi)}{n!}(q-\xi)^n$$

Thm: (Habiro) $T_\xi f = T_\xi g \Rightarrow f = g$

$$\widehat{\mathbb{Z}[q]} \hookrightarrow \mathbb{Z}[[1-q]]$$

integral domain

- $ev_\xi f$ for $\operatorname{ord} \xi = p, p^2, p^3, \ldots$

determine $f$

(2) $b = \prod_{i=1}^{s} p_i$ ~~odd~~ $S \subset \mathbb{N}$ $\Phi_n(q) = \prod_{(i,n)=1} (q - \xi_n^i)$

$\Phi_S = \{ \Phi_n(q) \mid n \in S \}$ $\Phi_S^*$ multiplicative set generated by $\Phi_S$

$$\mathbb{Z}[q]^S = \varprojlim_{f \in \Phi_S^*} \frac{\mathbb{Z}[q]}{f(q)}$$

$N_b = \{ n \in \mathbb{N} \mid (n, b$

$p^k = p_1^{k_1} \cdots p_s^{k_s}$

Example: $\mathbb{Z}[q]^{\mathbb{N}} = \widehat{\mathbb{Z}[q]}$

$$R_b := \mathbb{Z}\left[\tfrac{1}{b}\right][q]^{N_{odd}} = \prod_{k_i} \mathbb{Z}\left[\tfrac{1}{b}\right][q]^{p_1^{k_1} \cdots p_2^{k_2} N_b}$$

If $b$ is ~~even~~ odd? $\mathbb{Z}\left[\frac{1}{2b}\right]$

Thm (B-Le)

$$\mathbb{Z}\left[\tfrac{1}{b}\right][q]^{p^k N_b} \hookrightarrow \mathbb{Z}\left[\tfrac{1}{b}\right][[1 - q^{p^k}]]$$

Corollary:

- $\tau_M(\xi) \in \mathbb{Z}[\xi]$ $\forall \xi$ $\forall M$
  For $M \in N_b$
- $LMO_M$ determines $\{ \tau_M(\xi) \mid (\text{ord}\, \xi, b) = 1 \}$
- $\{ \tau_M(\xi) \}$ is determined by $\bigcup_{c \mid b} \{ \tau_M(\xi) \mid \text{ord}\, \xi = c\,n,\ n \in N_b \}$

$\exists$ generalized Ohtsuki series $\mathbb{Z}\left[\frac{1}{2b}\right][[1 - q^c]]$ dominate

## (3) Proofs $\qquad M = S^3(K_b)$

Idea: make ~~the definition~~ of $\tau_M(\xi)$ be independe[nt] on ord $\xi$

$$F_K(\xi) = \sum_{n=1}^{2r} [n]\, J_K[n] \qquad\qquad [n]^2 \sim (1-q^{-n})(1-$$

$$J'_{K_b}(n) = \sum C_{K,k}(q)\, (q^{1+n})_k\, (q^{1-n})_k$$

$$(q^a)_b = (1-q^a)\dots(1-q^{a+b-1})$$

$$F_K(\xi) = \sum_{k=0}^{\infty} C_{K,k}(q) \sum_{k=1}^{2r} q^{b\frac{n^2-1}{4}}\, (q^n)_{k+1}\, (q^{-n})_{k+1}$$

$$q^{an}$$

Lemma: $(r, b) = c$

$$\sum_{n=0}^{r} q^{b\frac{n^2-1}{4}}\, q^{an} = \begin{cases} 0 & c \nmid a \\ q^{-a^2/b} \sum_{n=0}^{r} q^{b\frac{n^2-1}{4}} & \end{cases}$$

Laplace transform:

$$\mathcal{L}_{b,c;n} : \mathbb{Z}[q^{\pm 1}, q^{\pm n}] \to \mathbb{Z}[q^{\pm 1/b}]$$

$$q^{an} \mapsto \begin{cases} 0 & c \nmid a \\ q^{-a^2/b} & \text{otherwise} \end{cases}$$

$$F_K(\xi) = \sum C_{K,k}(q)\, \underbrace{\mathcal{L}_{b,c;n}\left( (q^n)_{k+1}\, (q^{-n})_{k+1} \right)}\ \cdot \text{Gau[ss] sum}$$

Rogers - Ramanujan Identität

$$\prod_{k\geq 1} \frac{1}{(1-x^{5k-4})(1-x^{5k-1})} = \sum_{n\geq 0} \frac{x^{n^2}}{(1-x)(1-x^2)\ldots(1-x^n)}$$

MacMahon Interpret: $\lambda = (\lambda_1, \ldots \lambda_n)$ $\lambda_i \geq \lambda_{i+1}$ $\sum \lambda_i = n$

$\#\lambda$ with $\lambda_i = 1$ or $4$ mod $5$ $=$ $\#\lambda$ with $\lambda_i - \lambda_{i+1} \geq$ $\forall i$

$b = 1$

$$\mathcal{Z}_{1,n}\left((q^n)_{k+1} (q^{-n})_{k+1}\right) = 2\,(q^{k+1})_{k+1}$$

Example: $M = S^3(3_{1,+1})$ Poincaré sphere

$$I_M = \frac{q}{1-q} \sum_{k=0}^{\infty} (-1)^k q^{-\frac{(k+2)(3k+1)}{2}} (q^{k+1})_{k+1}$$

$$I_{S^3(4_1,1)} = \frac{q}{1-q} \sum_{k=0}^{\infty} (-1)^k q^{-(k+1)^2} (q^{k+1})_{k+1}$$

Question: ~~What is~~ Can homology theory whose Euler character. is $I_M$ ?

for Kirby 80th birthday