Categorification of quantum groups

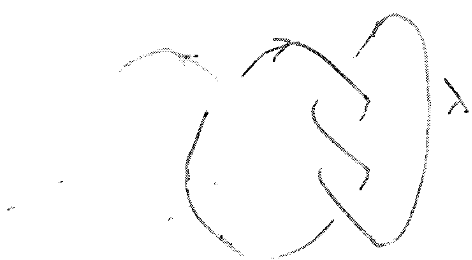
A. Lauda
M.K.

$\mathfrak{g}$ simple L.A. $\rightsquigarrow$ $U(\mathfrak{g})$ universal enveloping algebra $\rightsquigarrow$

$U_q(\mathfrak{g})$ quantum deformation

V_λ irrep of $U_q(\mathfrak{g})$ $\quad \lambda \in X_+$ positive integral weight

L link in $\mathbb{R}^3$, components colored by X_+



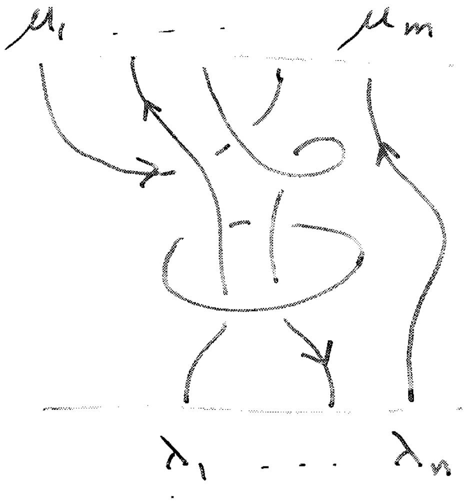
$P_{\underline{\lambda}}(L) \in \mathbb{Z}[q, q^{-1}]$

Reshetikhin-Turaev invariant

$\mathfrak{g} = \mathfrak{sl}(2)$ Jones polynomial, colored Jones polynomial

$\mathfrak{g} = \mathfrak{sl}(n)$, fund. rep $\to$ specializations of HOMFLYPT polynomial

Extends to tangles



$$\begin{array}{ll}
V_{\mu_1} \otimes \cdots \otimes V_{\mu_m} \circlearrowleft & \mathcal{U}_q(\mathfrak{g}) \\
\Big\uparrow f(T) & \text{RT invariant} \\
V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \circlearrowleft & \mathcal{U}_q(\mathfrak{g})
\end{array}$$

$f(T)$ intertwines $\mathcal{U}_q(\mathfrak{g})$ actions

Functor

$$\mathfrak{g}\text{-colored tangles} \longrightarrow \mathcal{U}_q(\mathfrak{g})\text{-modules}$$

Categorification

$H_{\underline{\lambda}}(L)$ bigraded homology theory of links

$$\chi(H_{\underline{\lambda}}(L)) = P_{\underline{\lambda}}(L)$$

Euler characteristic = Reshetikhin-Turaev invariant

functorial (link cobordisms)

$$V_{\underline{\lambda}} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$$

$C_{\underline{\lambda}}$ - triangulated category. Grothendieck group

$$K_0(C_{\underline{\lambda}}) \otimes \mathbb{C} = V_{\underline{\lambda}}$$

tangle $T \longrightarrow$ exact functor $F(T)$

$$\begin{tikzcd}
C_{\underline{\mu}} \arrow[r, "K_0"] & V_{\underline{\mu}} \\
C_{\underline{\lambda}} \arrow[u, "F(T)"] \arrow[r, "K_0"] & V_{\underline{\lambda}} \arrow[u, "f(T)"']
\end{tikzcd}$$

tangle cobordism $\longrightarrow$ natural transformation

Zheng, Categorification of tensor products, arxiv

$U_q(\mathfrak{g})$ acts on $V_{\underline{\lambda}}$, action intertwines $f(T)$,

invariants of tangles T

Categorified $U_q(\mathfrak{g})$ should act on $\mathcal{C}_{\underline{\lambda}}$

E_i, F_i - generators of $U_q(\mathfrak{g})$ $\longrightarrow$ $\mathcal{E}_i, \mathcal{F}_i$ - functors, natural transformations between their products.

1) Categorification of $U^+ \subset U_q(\mathfrak{g})$

only E_i's (no F_i's)

$$\mathfrak{g} = \mathfrak{sl}_n \qquad E_i = e_{i,i+1} = \begin{array}{cc} & i+1 \\ i & \begin{pmatrix} & 0 & \vdots & \\ - & - & 1 & \\ & 0 & & 0 \end{pmatrix} \end{array}$$

$$[E_i, E_j] = 0 \qquad |i-j| > 1$$

$$[E_i, [E_i, E_j]] = 0 \qquad |i-j| = 1$$

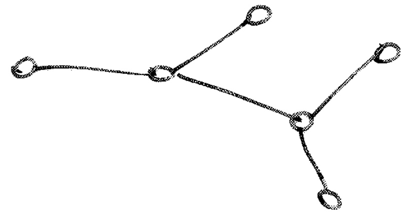
$$\begin{cases} E_i E_j = E_j E_i & |i-j| > 1 \\ 2 E_i E_j E_i = E_i^2 E_j + E_j E_i^2 & j = i \pm 1 \end{cases}$$

add q

$$E_i E_j = E_j E_i \qquad |i-j| > 1$$

$$(q + q^{-1}) E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \qquad j = i \pm 1$$

Γ unoriented graph



I - set of vertices

$\mathcal{U}^+ = \mathcal{U}^+(\Gamma)$

generators $E_i \quad i \in I$

relations $\quad E_i E_j = E_j E_i$

$$(q+q^{-1}) E_i E_j E_i = E_i^2 E_j + E_j E_i^2$$

2nd divided power $\quad E_i^{(2)} = \dfrac{E_i^2}{q+q^{-1}}$

$$E_i E_j E_i = E_i^{(2)} E_j + E_j E_i^{(2)}$$

Functors $\mathcal{E}_i$ and their products

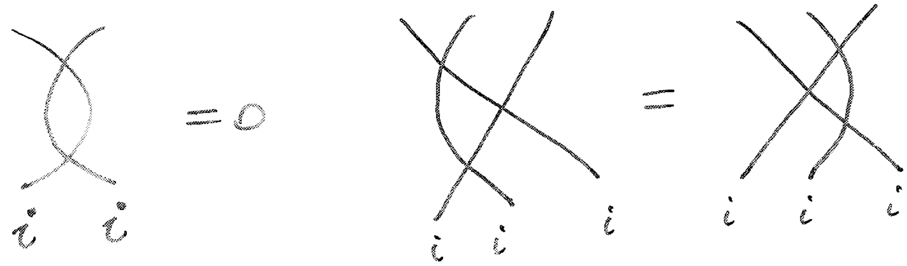
$\mathcal{E}_{i_1} \dots \mathcal{E}_{i_m}$. Hom $(\mathcal{E}_{i_1} \dots \mathcal{E}_{i_m}, \mathcal{E}_{j_1} \dots \mathcal{E}_{j_m})$

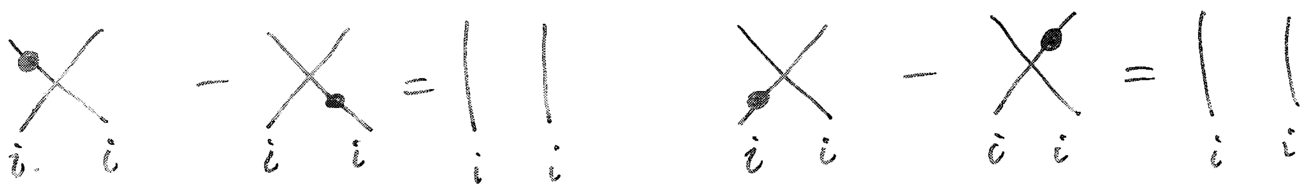
will be given by pictures

braid - like diagrams with dots, strands labelled by vertices of the graph

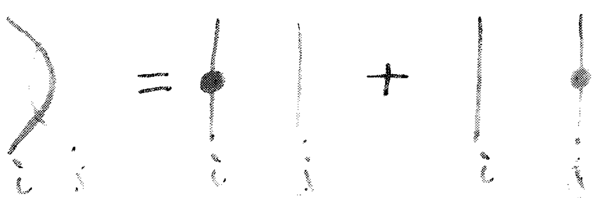
Composition of natural transformations is given by concatenation of diagrams

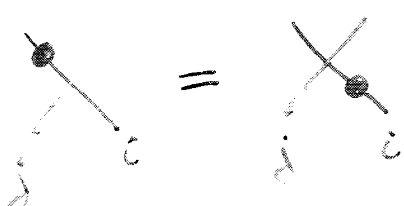
Relations:

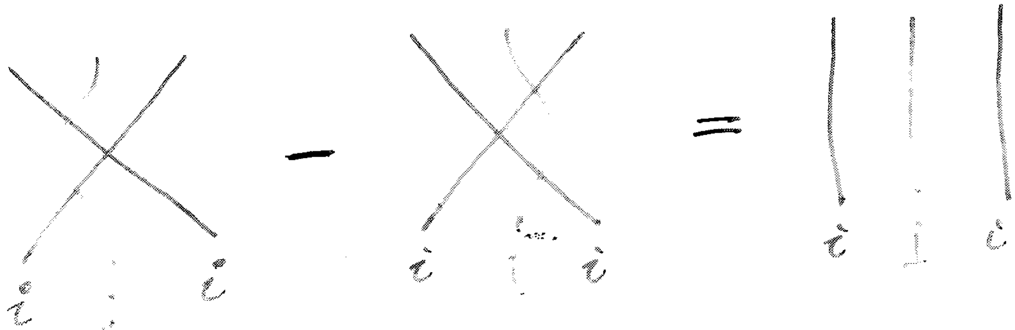
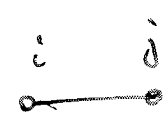


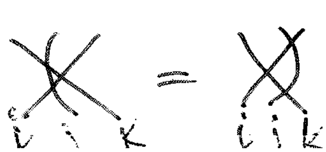


 if 

 if 

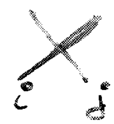
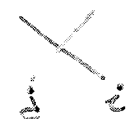
 if $i \neq j$

 if 

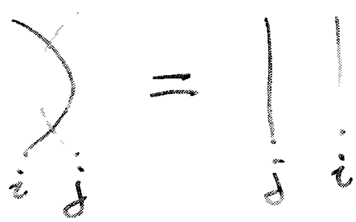
otherwise 

1) $E_i E_j = E_j E_i$ if $\overset{i}{\circ} \quad \overset{j}{\circ}$

$\mathcal{E}_i \mathcal{E}_j \simeq \mathcal{E}_j \mathcal{E}_i$ if $\overset{i}{\circ} \quad \overset{j}{\circ}$

$$\mathcal{E}_i \mathcal{E}_j \longrightarrow \mathcal{E}_j \mathcal{E}_i \longrightarrow \mathcal{E}_i \mathcal{E}_j$$

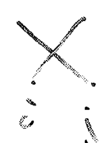


← diagram of the identity functor

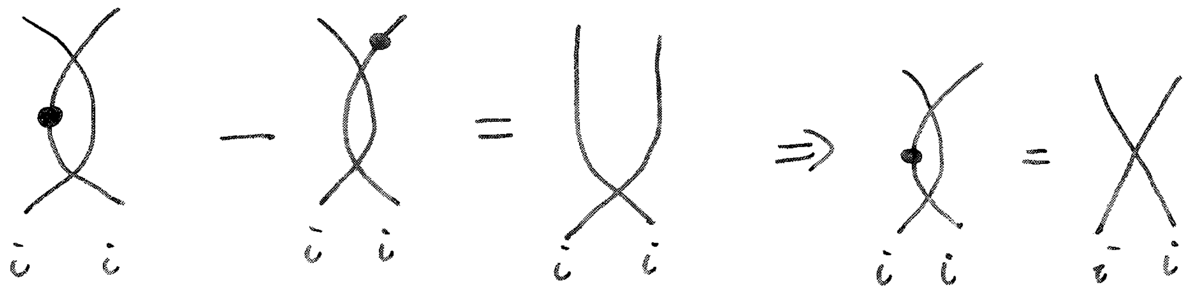
2) $E_i^{(2)} = \dfrac{E_i^2}{q+q^{-1}}$ $\quad q \longrightarrow$ grading shift

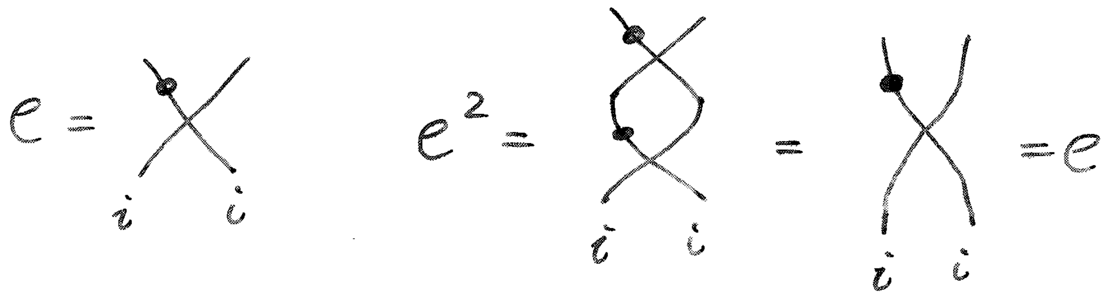
$E_i^2 = (q+q^{-1}) E_i^{(2)}$

deg  $= 2$

deg  $= \begin{cases} -2 & i=j \\ 0 & \overset{i}{\circ} \quad \overset{j}{\circ} \\ 1 & \underset{i}{\circ}\!\!-\!\!\!-\!\!\!-\underset{j}{\circ} \end{cases}$

$$\mathcal{E}_i \mathcal{E}_i \simeq \mathcal{E}_i^{(2)}\{1\} \oplus \mathcal{E}_i^{(2)}\{-1\}.$$





e idempotent in $\mathrm{End}(\mathcal{E}_i^2)$

Define $\mathcal{E}_i^{(2)}$ as projection onto this idempotent.

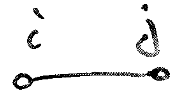
If R is a ring, e an idempotent in R, then

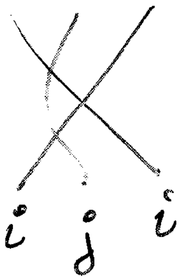
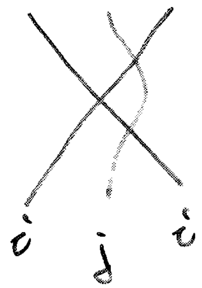
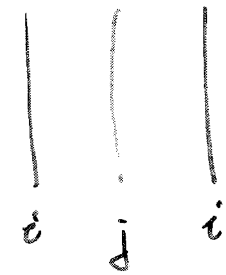
$R \simeq Re \oplus R(1-e)$ as left R-modules

$$\mathcal{E}_i^2 \simeq \mathcal{E}_i^2 e \oplus \mathcal{E}_i^2(1-e)$$

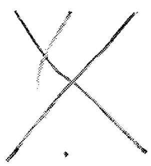
$\mathcal{E}_i^2 e \simeq \mathcal{E}_i^2(1-e)$ up to grading shift

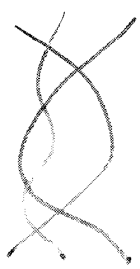
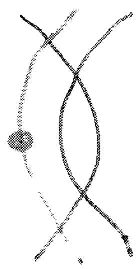
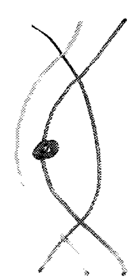
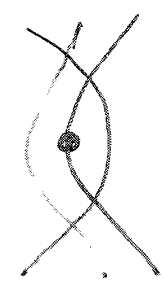
$$\mathcal{E}_i^2 \simeq \mathcal{E}_i^{(2)} \oplus \mathcal{E}_i^{(2)}, \text{ where } \mathcal{E}_i^{(2)} = \mathcal{E}_i^2 e$$

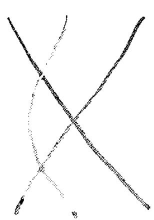
$E_i E_j E_i = E_i^{(2)} E_j + E_j E_i^{(2)}$ if 

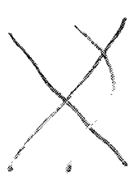
 $-$  $=$ 

identity endomorphism of $\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i$

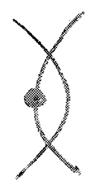
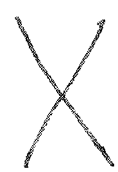
$e' =$ 

$(e')^2 =$  $=$  $+$  $=$  $=$

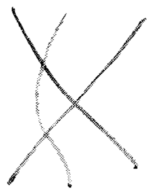
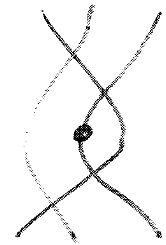
 $= e'$ $\Rightarrow$ e' is an idempotent

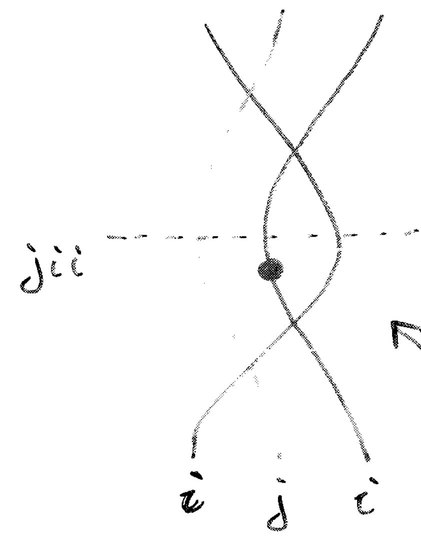
$e'' = -$  $(e'')^2 = e''$ $e'e'' = e''e' = 0$

$1_{iji} = e' + e'' \Rightarrow$

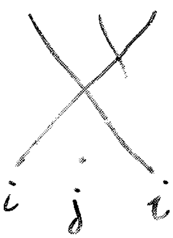
$\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i \simeq \mathcal{E}_i \mathcal{E}_j \mathcal{E}_i e' \oplus \mathcal{E}_i \mathcal{E}_j \mathcal{E}_i e''$  $=$ 

$$\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i = \underset{\mathcal{E}_j \mathcal{E}_i^{(2)}}{\underset{\wr\!\shortparallel}{\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i e'}} \oplus \underset{\mathcal{E}_i^{(2)} \mathcal{E}_j}{\underset{\wr\!\shortparallel}{\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i e''}}$$

$e' =$  $=$ 



idempotent e of projection onto $\mathcal{E}_i^{(2)}$

$e'' = -$ 

$$\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i \simeq \mathcal{E}_j \mathcal{E}_i^{(2)} \oplus \mathcal{E}_i^{(2)} \mathcal{E}_j$$

<u>Theorem</u>

$\mathcal{U}^+ \simeq$ Grothendieck group of these endomorphism rings

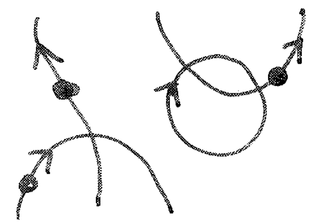
A. Lauda, M.K. arXiv 0803.4121

Categorification of quantum $sl(2)$

$\Gamma = \circ$

A. Lauda

all lines have the same label i



$$U_q(sl_2)$$

$E, F, K^{\pm 1}$

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

Any fin-dim rep. V has a weight decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V(n)$$

$$\begin{tikzcd} V(n+2) \arrow[d, bend left, "F"] \\ V(n) \arrow[u, bend left, "E"] \arrow[d, bend left, "F"] \\ V(n-2) \arrow[u, bend left, "E"] \end{tikzcd}$$

$v \in V(n) \Leftrightarrow Kv = q^n v$

add to $U_q(sl_2)$ operators 1_n of projection onto $V(n)$

$$U_q(sl_2) \longrightarrow \dot{U}(sl_2)$$

$1 \longrightarrow$ Collection of idempotents 1_n, $n \in \mathbb{Z}$

$K 1_n = q^n 1_n \quad \to$ K vanishes

$$E 1_n = 1_{n+2} E = 1_{n+2} E 1_n$$

$$F 1_n = 1_{n-2} F = 1_{n-2} F 1_n$$

$$(EF - FE) 1_n = [n] 1_n \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$\dot{U}(sl_2)$ has a basis $\{E^a F^b 1_n\}$ $n \in \mathbb{Z}$, $a, b \geq 0$

non-unital ring.

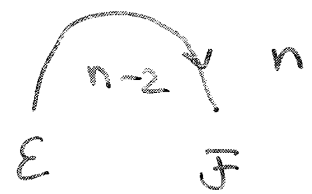
$\dot{U}(sl_2)$, $\dot{U}(sl_n)$ Beilinson-Lusztig-MacPherson

$\dot{U}(\mathfrak{g})$ Lusztig

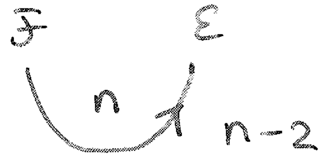
After categorification, E and F become biadjoint functors $\mathcal{E}$ and $\mathcal{F}$

$$\begin{tikzcd} n \arrow[r, bend left, "\mathcal{E}"] & n+2 \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

$n \xrightarrow{\mathbb{1}_n} n$
identity functor

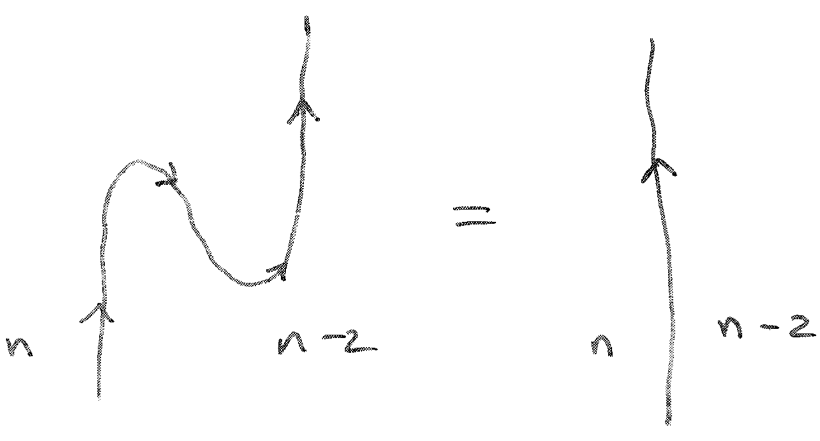


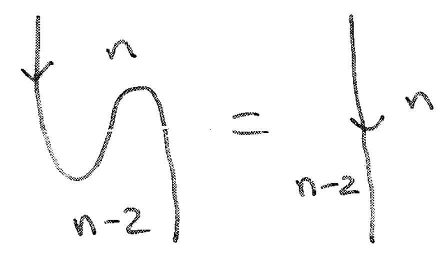
$\mathcal{E}\mathcal{F}\mathbb{1}_n \longrightarrow \mathbb{1}_n$



$\mathbb{1}_{n-2} \longrightarrow \mathcal{F}\mathcal{E}\mathbb{1}_{n-2}$

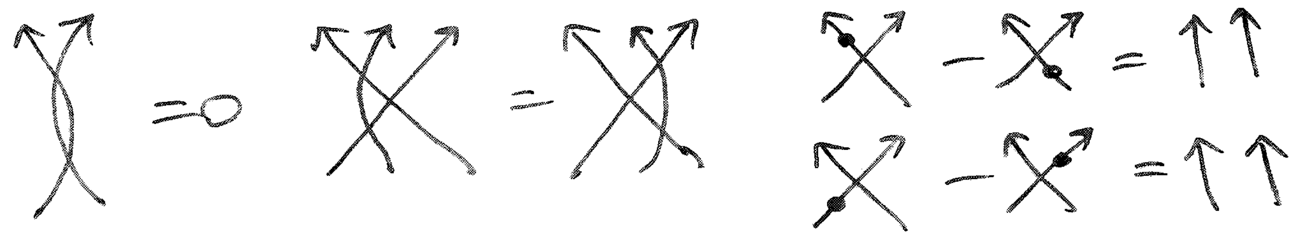
$\mathcal{E}$ is a left adjoint of $\mathcal{F}$

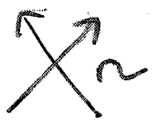




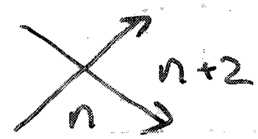
+ opposite orientations $\longrightarrow$ biadjointness.

Recall relations in categorified $U^+(\mathfrak{sl}_2)$

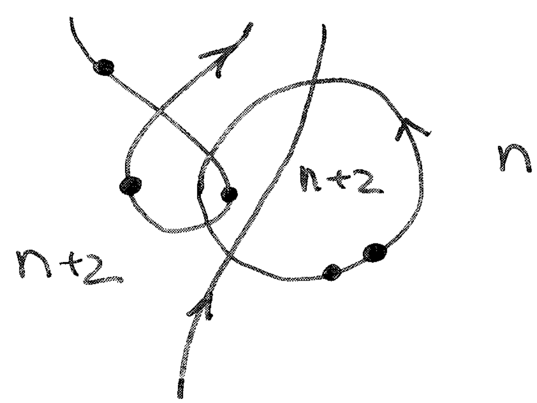




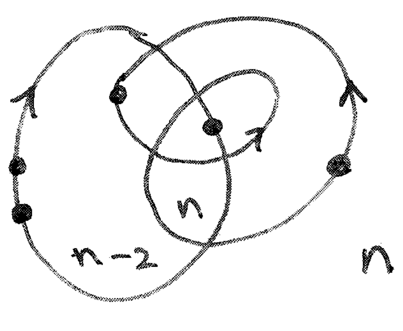
$$\begin{array}{c} \mathcal{E}^2\mathbb{1}_n \\ \uparrow \\ \mathcal{E}^2\mathbb{1}_n \end{array}$$



$$\begin{array}{c} \mathcal{F}\mathcal{E}\mathbb{1}_{n+2} \\ \uparrow \\ \mathcal{E}\mathcal{F}\mathbb{1}_{n+2} \end{array}$$



$$\begin{array}{c} \mathcal{F}\mathcal{E}\mathcal{E}\mathbb{1}_n \\ \uparrow \\ \mathcal{E}\mathbb{1}_n \end{array}$$



$$\begin{array}{c} \mathbb{1}_n \\ \uparrow \\ \mathbb{1}_n \end{array}$$

· Closed diagrams give elements of $\mathrm{End}(\mathbb{1}_n)$

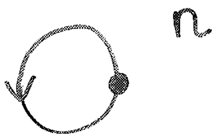
Degree of a diagram

							
deg	2	-2	$n+1$	$1-n$	$n+1$	$1-n$	0

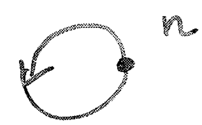
allow arbitrary isotopies of diagrams

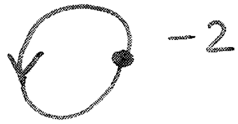
all relations are homogeneous

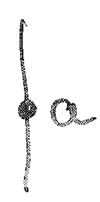
A closed diagram of negative degree $=0$



$\deg = 2(n+1)+2 = 2n+4$

if $n<-2$,  $=0$

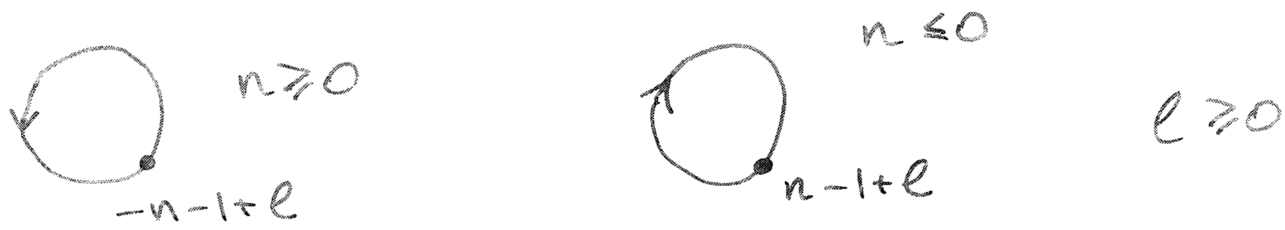
$n=-2$  $=$ -2

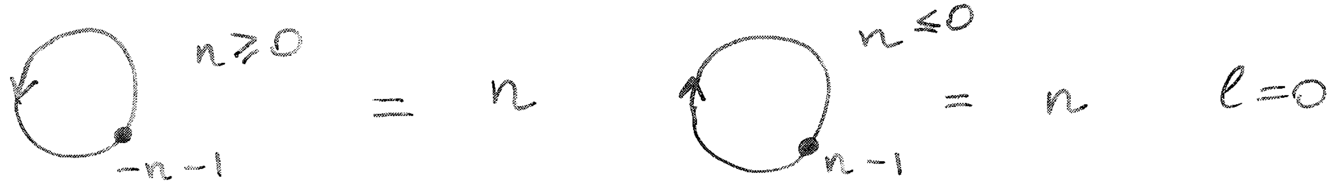
 $=$ 

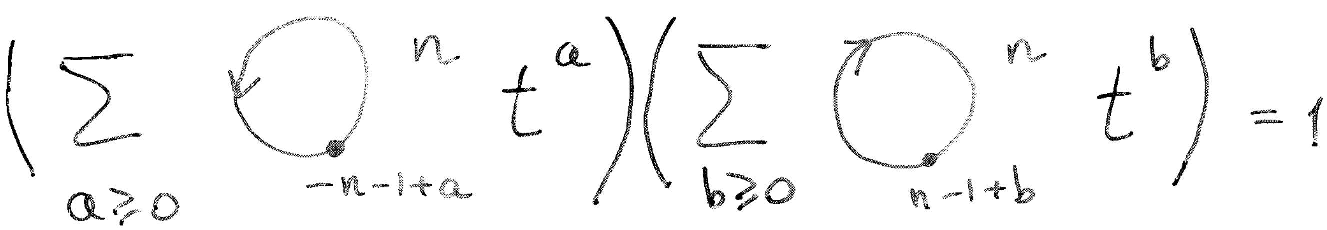
To write the rest of the defining relations in a convenient form, introduce <u>fake bubbles</u>

$\deg \geq 0$, $a<0$

fake bubbles







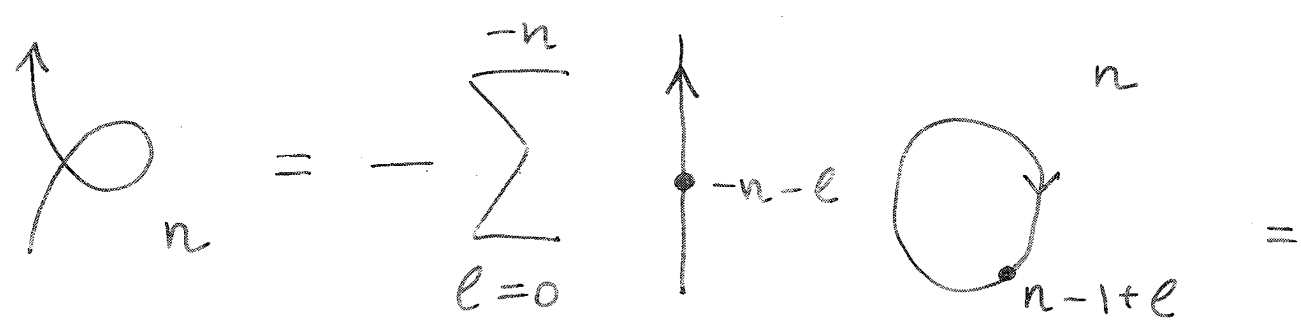
analogous to defining relations in $H^*(Gr(\infty, \infty))$

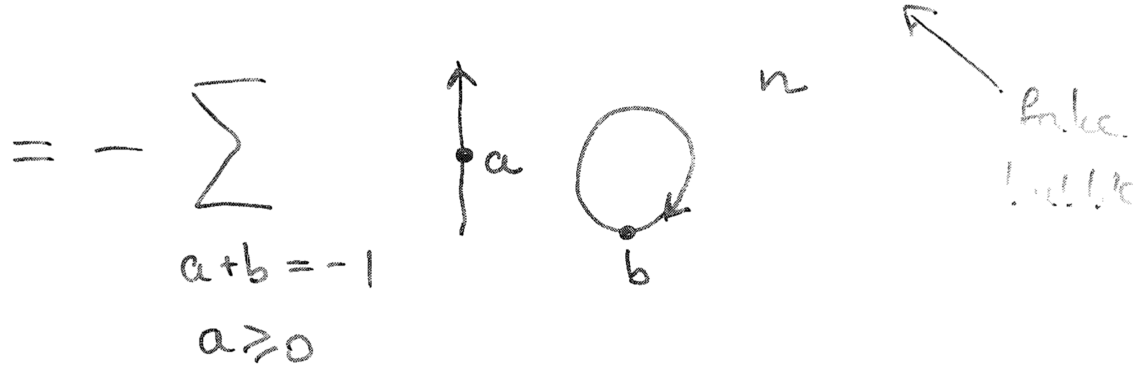
$\lim Gr(m, m)$

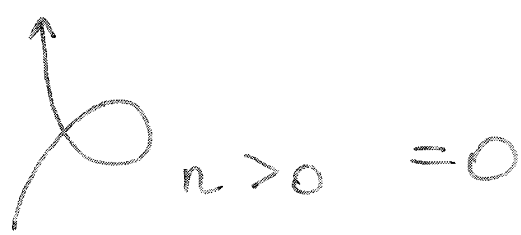
$\mathbb{C}^m \subset \mathbb{C}^{2m}$

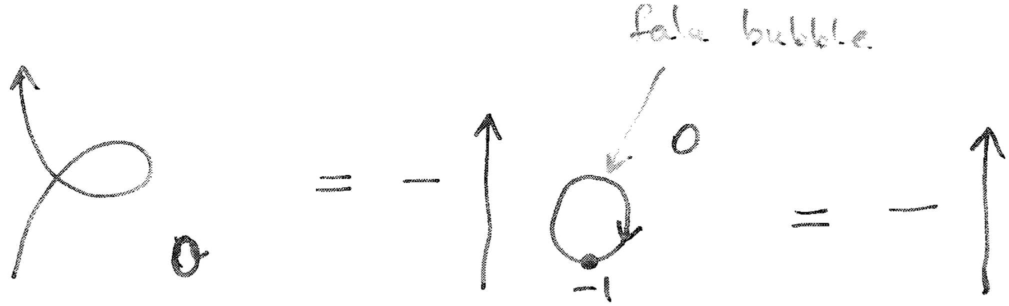
$$(1 + x_1 t + x_2 t^2 + \ldots)(1 + y_1 t + y_2 t^2 + \ldots) = 1$$

$$x_1 + y_1 = 0, \quad x_2 + x_1 y_1 + y_2 = 0, \ldots$$

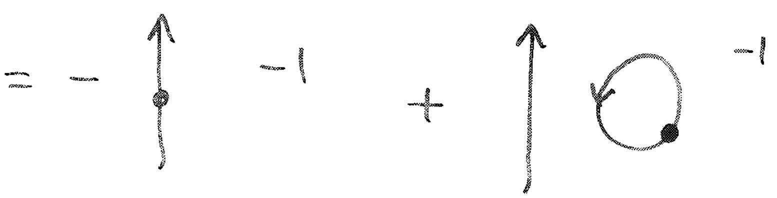

n
= −
−n
l=0
−n−l
n
n−1+l
=

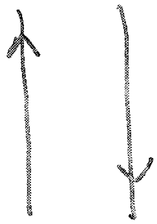

= −
a+b=−1
a≥0
a
n
b
fake bubble


n > 0
= 0

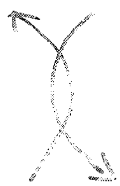
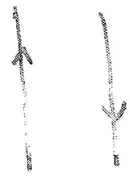

fake bubble
0
= −
0
−1
= −

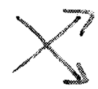

−1
= −
1
−1
−2
−
−1
−1
=

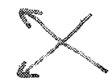

= −
−1
+
−1

 $_n$ $= -$  $_n$ $+ \displaystyle\sum_{\substack{a+b+c=-2 \\ a,b\geq 0}}$  $_n$

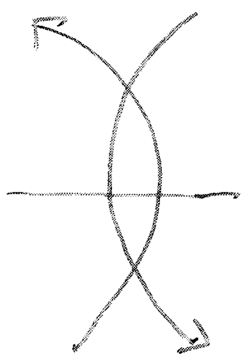
if $n \leq 0$

 $= -$ 





$$\mathrm{Id}_{\mathcal{E}\mathcal{F}\mathbb{1}_n} = -\left(\mathcal{E}\mathcal{F}\mathbb{1}_n \longrightarrow \mathcal{F}\mathcal{E}\mathbb{1}_n \longrightarrow \mathcal{E}\mathcal{F}\mathbb{1}_n \right)$$



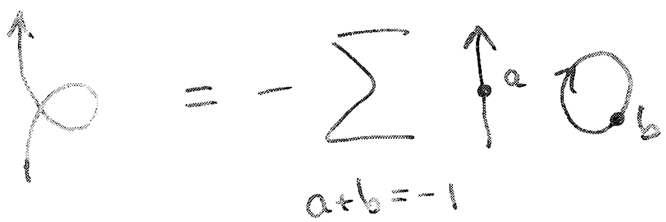
$$\begin{tikzcd} \mathcal{E}\mathcal{F}\mathbb{1}_n \\ \mathcal{F}\mathcal{E}\mathbb{1}_n \arrow[u] \\ \mathcal{E}\mathcal{F}\mathbb{1}_n \arrow[u] \arrow[uu, bend right=60, "-\mathrm{Id}"'] \end{tikzcd}$$

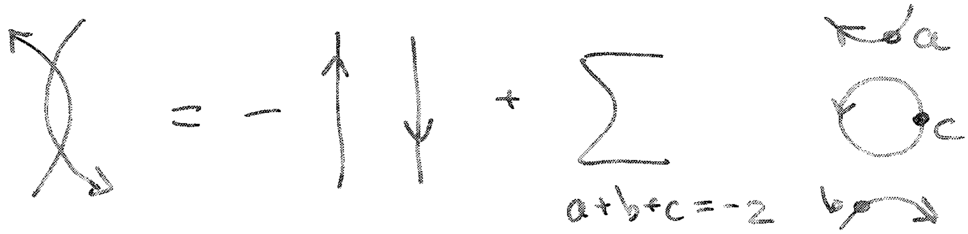
$$\mathcal{F}\mathcal{E}\mathbb{1}_n \simeq \mathcal{E}\mathcal{F}\mathbb{1}_n \oplus \mathbb{1}_n^{-[n]} \qquad n \leq 0$$

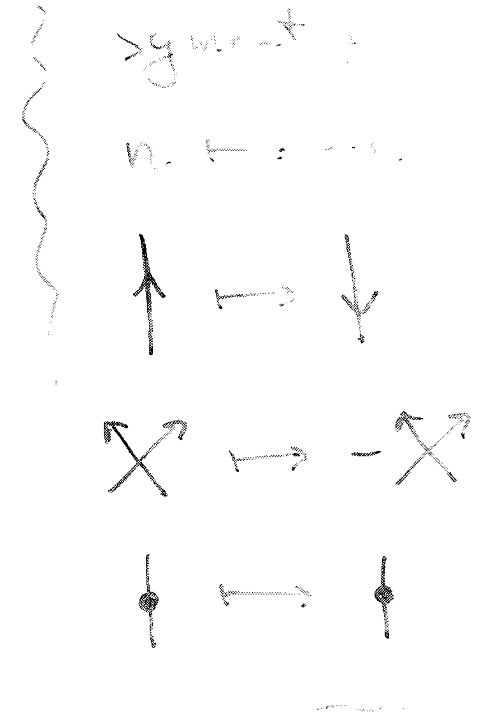
$$[E,F]1_n = [n]1_n$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

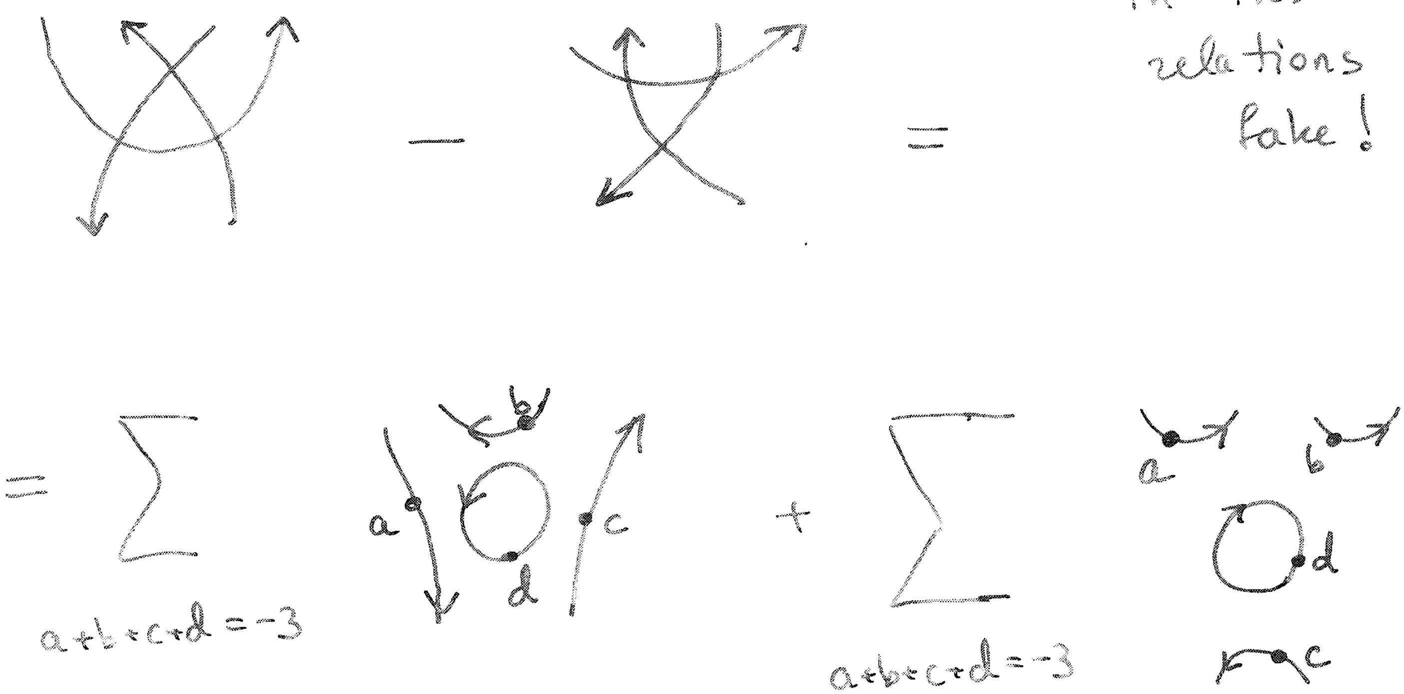
Reidemeister-like relations

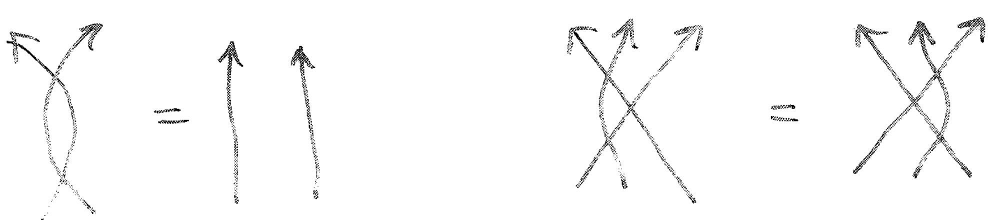






all bubbles in these relations are fake!





<u>Theorem</u> (Aaron Lauda, arxiv 0803.3652)

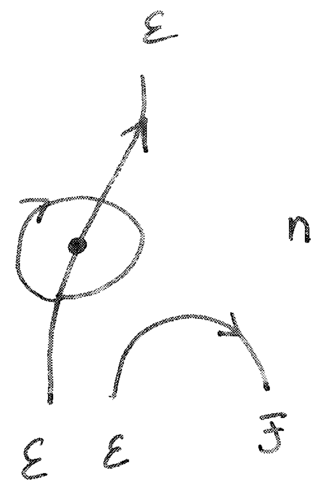
This graphical calculus is consistent and categorifies $\dot{\mathcal{U}}(sl_2)$.

$\dot{\mathcal{U}}(sl_2) \simeq$ Grothendieck ring / category of this 2-category.

Objects: $n \in \mathbb{Z}$

1-morphisms: $m \quad \overset{+}{\underset{\mathcal{E}}{\bullet}} \quad \overset{+}{\underset{\mathcal{E}}{\bullet}} \quad \overset{-}{\underset{\mathcal{F}}{\bullet}} \quad n$

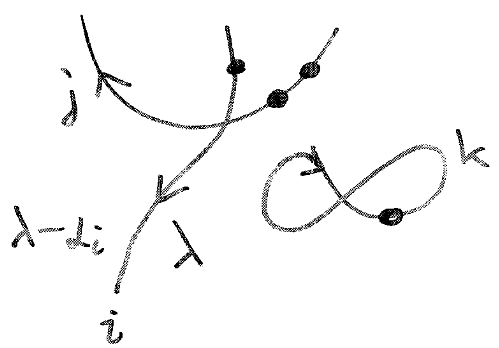
2-morphisms:



Categorification of $\mathring{\mathcal{U}}(sl_2)$ + Categorification of U^+ for any Kac-Moody Lie algebra

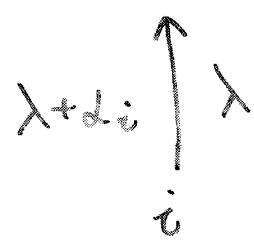
$$\Big\downarrow$$

Categorification of $\mathring{\mathcal{U}}(sl_n)$



regions are labelled by integral weights of sl_n

$\lambda \in X$



strands are labelled by simple roots

+ dots on strands

+ local relations.

<u>Theorem</u>: Grothendieck ring of this 2-category is $\mathring{\mathcal{U}}(sl_n)$.

For any Kac-Moody Lie algebra there is a surjective homomorphism

$$\dot{\mathcal{U}}(\mathfrak{g}) \twoheadrightarrow K_0$$

Grothendieck group

To show that $\twoheadrightarrow$ is an isomorphism, it suffices to check that the graphical calculus is nondegenerate (that obvious spanning sets in $\mathrm{Hom}(\mathcal{E}_{i_1}\mathcal{F}_{j_1}\cdots \mathbb{1}_n, \mathcal{E}_{i_1'}\mathcal{F}_{j_1'}\cdots \mathbb{1}_n)$ are bases).

A special case of nongeneracy in $\mathfrak{sl}(2)$ case:

For $n \geq 0$, $\mathrm{Hom}(\mathbb{1}_n, \mathbb{1}_n)$ is a polynomial algebra on generators