$4-$manifolds and the A-B slice problem

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History and motivation:

Geometric classification tools in higher dimensions:

Surgery: Given an $n-$dimensional Poincaré complex X, is there an $n-$manifold M^n homotopy equivalent to it?

s-cobordism theorem: Given an $(n+1)$-dimensional s-cobordism W with $\partial W = M_1 \sqcup (-M_2)$, is W isomorphic to the product $M_1 \times [0,1]$?

In dimension $n = 4$: *smoothly* both surgery and s-cobordism fail even in the simply-connected case (Donaldson)

Dimension $n = 4$, topological category:

M. Freedman (1982): Both surgery and s-cobordism conjectures hold for $\pi_1 = 1$ and more generally for elementary amenable groups.

Applications:

- Classification of topological simply-connected $4-$manifolds.
- Slice results for knots and links, in particular: Alexander polynomial 1 knots are slice.
- (F. Quinn): Classification of homeomorphisms (up to isotopy) of simply-connected $4-$manifolds.

More applications:

- Classification of $4-$manifolds with $\pi_1 \cong \mathbb{Z}$ (Freedman-Quinn)

- Classification for π_1 finite cyclic (oriented case: Hambleton-Kreck)

More recently:

- Classification for Boumslag-Solitar groups (Hambleton-Kreck-Teichner)

- New examples of slice knots (Friedl-Teichner)

Currently the class of good groups, for which surgery and the s-cobordism conjectures are known to hold, includes the groups of subexponential growth, and is closed under extensions and direct limits. (Freedman-Teichner 1995, K.-Quinn 2000)

Amenable groups?

Conjecture (Freedman 1983) Surgery fails for free groups.

More specifically, there does not exist a topological $4-$manifold M, homotopy equivalent to $\vee^3 S^1$, with $\partial M = \mathcal{S}_0(Wh(Bor))$.

Equivalently: The Whitehead double of the Borromean rings is not a “free” slice link.

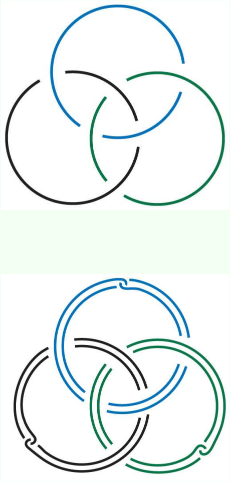


Figure: The untwisted Whitehead double of the Borromean rings.

A *decomposition* of D^4, $D^4 = A \cup B$, is an extension to the $4-$ball of the standard genus one Heegaard decomposition of the $3-$sphere. Specified distinguished curves $\alpha \subset \partial A, \beta \subset \partial B$ form the Hopf link in $S^3 = \partial D^4$.

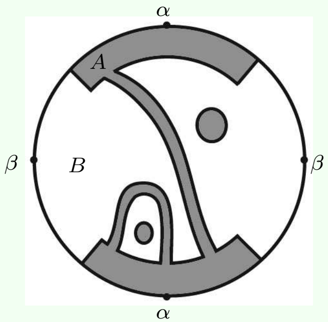


Figure: A $2-$dimensional example of a decomposition, $D^2 = A \cup B$.

An $n-$component link $L \subset S^3$ is *weakly $A - B$ slice* if there exist decompositions $(A_i, B_i), i = 1, \ldots, n$ of D^4 and disjoint embeddings of all $2n$ manifolds $\{A_i, B_i\}$ into D^4 so that the distinguished curves $(\alpha_1, \ldots, \alpha_n)$ form the link L, and the curves $(\beta_1, \ldots, \beta_n)$ form a parallel copy of L.

L is A-B slice if, in addition, the new embeddings $A_i \subset D^4, B_i \subset D^4$ are standard: isotopic to the original embeddings.

Connection with the surgery conjecture:

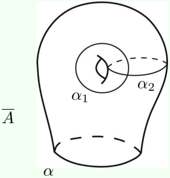
Topological $4-$dimensional surgery works for all groups if and only if the Borromean rings (and a certain family of their generalizations) are A-B slice.

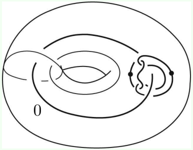
Outline: Suppose the existence of M^4, homotopy equivalent to $\vee^3 S^1$, with $\partial M = \mathcal{S}_0(Wh(Bor))$. Its universal cover $\widetilde{M}$ is contractible. The end-point compactification of $\widetilde{M}$ is homeomorphic to the $4-$ball (Freedman). $\pi_1(M)$, the free group on three generators, acts on D^4.

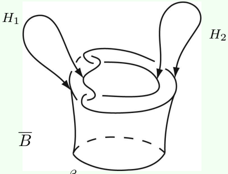
Freedman's conjecture: The Borromean rings are not A-B slice. (Stronger version: not even weakly A-B slice.)

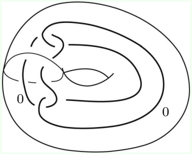
Theorem (K, 2006) The Borromean rings are weakly A-B slice.

Outline of the proof: Consider a preliminary decomposition $D^4 = \overline{A} \cup \overline{B}$ (a schematic "spine" picture is on the left, and a precise Kirby handle diagram is given on the right):

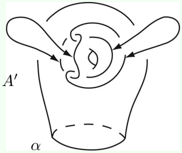


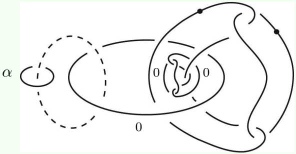


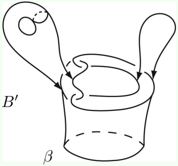


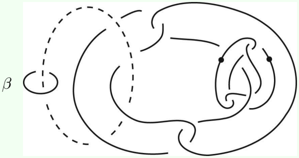


Another preliminary decomposition $D^4 = A' \cup B'$:









The decompositions $D^4 = \overline{A} \cup \overline{B} = A' \cup B'$ above are examples of *model decompositions* (introduced by M. Freedman and X.-S. Lin):

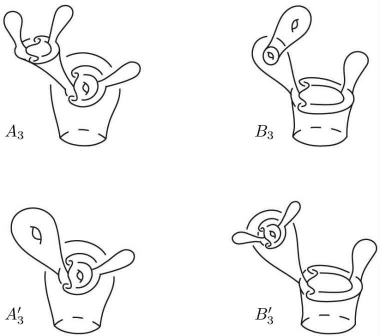
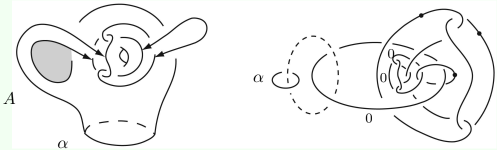
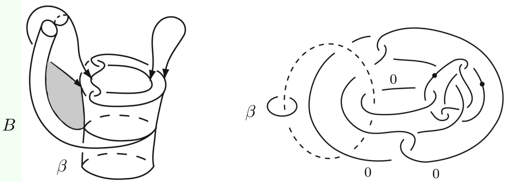


Figure: Examples of model decompositions of height 3.

Theorem (K.) The Borromean rings are not A-B slice (not even weakly A-B slice) when restricted to the class of model decompositions.

Finally, consider the decomposition $D^4 = A \cup B$:





Claim: There exist disjoint embeddings of six manifolds into D^4: three copies $\{A_i\}$ of A and three copies $\{B_i\}$ of B, such that $\alpha_1, \alpha_2, \alpha_3$ form the Borromean rings; $\beta_1, \beta_2, \beta_3$ are a parallel copy. This proves that the Borromean rings are weakly A-B slice.

Proof of the claim: a “relative-slice” problem. An illustration in 2 dimensions:

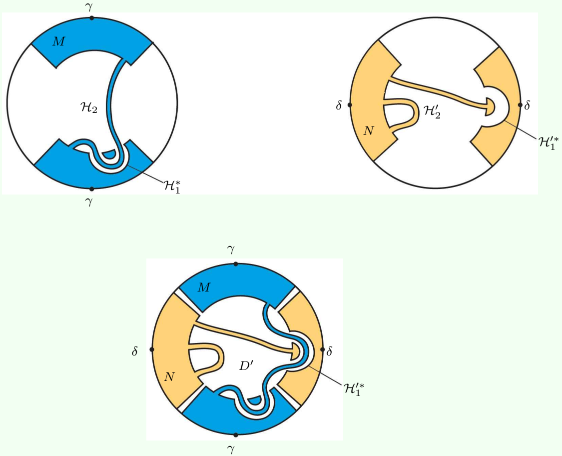


Figure: Disjoint embeddings of (M, γ), (N, δ) in (D^4, S^3), where γ, δ form a Hopf link in S^3.

There is a secondary obstruction, taking into account the embeddings $A \hookrightarrow D^4$, $B \hookrightarrow D^4$, showing that these decompositions do not solve the A-B slice problem.

The rest of this talk will concern more recent developments (Joint work with Michael Freedman):

Given a decomposition $D^4 = A \cup B$, consider the $3-$manifold $X^3 = A \cap B$ with torus boundary.

It seems reasonable to believe that X^3 together with

$$ker[\pi_1(X)/\pi_1^k(X) \longrightarrow \pi_1(A)/\pi_1^k(A)],$$

$$ker[\pi_1(X)/\pi_1^k(X) \longrightarrow \pi_1(B)/\pi_1^k(B)]$$

encode the relevant information about the decomposition, where π_1^k denotes the kth term of the lower central series.

Tools used for analyzing this problem:

- Nilpotent quotients, in particular the Milnor group.
- Massey products.
- Rational homotopy theory: Sullivan's minimal models.

These techniques are useful for working with specific decompositions, but a common problem is indeterminacy which makes it difficult to give a "uniform" analysis of all possible decompositions.

Consider $\mathcal{M} = \{(M, \gamma) | M$ is a codimension zero, smooth, compact submanifold of D^4, and $M \cap \partial D^4$ is a tubular neighborhood of an unknotted circle $\gamma \subset S^3\}$.

A **topological arbiter** is an invariant $\mathcal{A}\colon\ \mathcal{M} \longrightarrow \{0, 1\}$ satisfying axioms (1) – (3):

(1) "$\mathcal{A}$ is topological": If (M, γ) is ambiently isotopic to (M', γ') in D^4 then $\mathcal{A}(M, \gamma) = \mathcal{A}(M', \gamma')$.

(2) "Greedy axiom": If $(M, \gamma) \subset (M', \gamma)$ and $\mathcal{A}(M, \gamma) = 1$ then $\mathcal{A}(M', \gamma') = 1$.

(3) "Alexander duality": Let $D^4 = A \cup B$ be a decomposition of D^4, so the distinguished curves α, β of A, B form the Hopf link in ∂D^4. Then $\mathcal{A}(A, \alpha) + \mathcal{A}(B, \beta) = 1$.

Theorem. There are uncountably many topological arbiters (satisfying axioms (1)-(3)) on D^4.

Axiom (4): Suppose $\mathcal{A}(M', \gamma') = 1$ and $\mathcal{A}(M'', \gamma';) = 1$. Then $\mathcal{A}(D(M', M''), \gamma) = 1$ where $D(M', M'')$ is the “Bing double”.

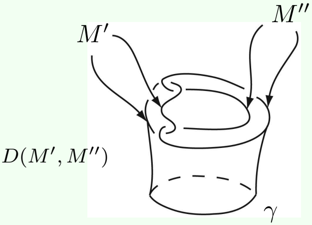


Figure: The Bing double of M', M''.

Proposition. A topological arbiter satisfying Axioms (1)-(4) is an obstruction to topological surgery.

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Outline of the proof of the theorem above: construction of a tree of submanifolds with pairwise non-embedding properties: