8/15/05

Seiberg-Witten equations &
dynamics of vector fields in dim 3
Clifford Taubes

M = 3-dim'l oriented manifold
Ω = volume

#1 $v \lrcorner \Omega = \omega_v$ $\quad d\omega_v$ $\quad$ Vector field dynamics

$\phi_t : M \longrightarrow M$, diffeomorphisms

$$\frac{d\phi}{dt} = v(\phi_t)$$

Contact 1-form $a \in \Omega^1(M)$

$\Omega = a \wedge da > 0$

$v \lrcorner da = 0$

$\cancel{\alpha} \lrcorner a = 1$

$\omega_v = da$

Thm: v has at least 1 closed orbit.

Stable Hamiltonian (joint w/ Hutchings)

$a \quad \omega_v \quad a \wedge \omega_v = \Omega$

$da = f\omega_v$

In this case, v has a closed orbit

except if $\begin{array}{c} M \supset T^2 \\ \downarrow \\ S^1 \end{array}$

What can be said about uniquely ergodic vector fields?

Say that a vector field is exact if $\omega_v = db$

asymptotic linking number:

$$\delta_v = \int_M b \wedge \omega_v$$

<u>Thm:</u> If v is uniquely ergodic, then $\delta_v = 0$.

<u>Example:</u>

Horocycle flow on unit tangent bundle to the surface

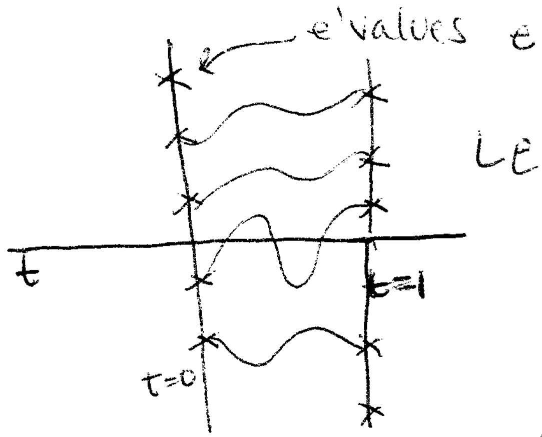
$$\omega_v = db$$

$$b \wedge db = 0$$

#2 Spectral Dynamics

$\mathbb{H}$ - Hilbert space

$L : \mathbb{H} \to \mathbb{H}$, self-adjoint



$Le = \lambda e$

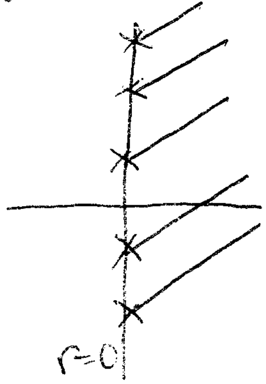
Spectral flow $\{L_t\}_{t \in [0,1]}$

<u>Simple</u> <u>example</u>:

$$L_r = i \frac{\partial}{\partial \theta} + r$$

$\{e^{in\theta}\}_{n \in \mathbb{Z}}$ $\lambda = -n + r$

e'values move linearly



$Sf = r + O(1)$
spectral flow

Fix metric on M^3 — Ω
$|v| = 1$ and $|w_v| = 1$

$$
\begin{tikzcd}
F \supseteq U(2) \supseteq U(1) \arrow[d] & \\
Fr \supseteq SO(3) \arrow[d] & \\
M &
\end{tikzcd}
$$

$\mathrm{Spin}^c \Longleftrightarrow H^2(M; \mathbb{Z})$

$F \times_{U(2)} \mathbb{C}^2 = \mathbb{S}$

$TM \xrightarrow{\phi} \mathrm{End}(\mathbb{S})$

$cl(b)\,cl(c) = -\langle b, c\rangle - cl(*(b \wedge c))$
Clifford multiplication

<u>Dirac</u> <u>operator</u>

$C^\infty(M, \mathbb{S}) \xrightarrow{\nabla_A} C^\infty(M; \mathbb{S} \otimes T^*M) \xrightarrow{cl} C^\infty(\mathbb{S}; M)$

D_A-irb

$v \lrcorner \Omega = w_v - db$

Spectral flow is independent of b if $c_1(\det \mathbb{S}) =$ torsion

Thm: $\delta f = \frac{-1}{8\pi^2} r^2 \int b \wedge w_v + \mathcal{O}(r^{2-\delta})$

$\delta > \frac{1}{32}$

$b \wedge db = 0$

#3 Seiberg-Witten dynamics

Spin^c structure A, a connection on $\det(\mathbb{S})$

ψ, a section of $\mathbb{S}$

- $D_A \psi = 0$
- $* F_A = r(\psi^+ \tau \psi - i * w_v)$

$\langle c, \eta^+ \tau \psi \rangle = \eta^+ cl(c) \psi$

Thm:
Suppose $\{(A_n, \psi_n), r_n\}_{n=1,2,\ldots}$
with $r_n \to \infty$

- $E(A_n) = i \int * w_v \wedge F_{A_n} < \varepsilon < \infty$
- $\sup_M 1 - |\psi_n| > \delta > 0$

then v has a closed orbit.

Suppose $E(A_n) \longrightarrow \infty$

Define $f \longrightarrow \dfrac{r \int f(1-|\psi_n|^2)}{E(A_n)}$

Then this converges as $n \to \infty$ to an invariant measure which is nontrivial if $A \neq 0$

Remarks:
If $c_1(\det S)$ is torsion, then there exists lots of $\{(A_n, \psi_n), r_n\}$ obeying SW-equations.

SW-Floer homology.
cycles come from solutions to SW equations

Irreducible soln: $C^\infty(M, S^1)$ on solutions
u

$$A \to A - 2u^{-1}du \qquad \psi \longrightarrow u\psi$$

stabilizer $= 1$, $\psi \neq 0$

Reducible soln: S^1, $\psi \equiv 0$
$A = \Gamma_{flat} - irb' \qquad db = \omega_r$

<u>Thm</u> (Kronheimer, Mrowka)
There are infinitely many non-trivial classes with degrees $\to -\infty$

~~Day~~ 3 [illegible] of v. field dynamics [illegible]

Part I: Configuration space dynamics:

- M = closed oriented [illegible] d mfld.
- V = vector field

V generates 1 param. family of diffeo.

$$x \to \phi_t(x)$$

$$\frac{d\phi_t}{dt} = V(\phi_t) \qquad \phi_0 = x$$

Interest is in behavior of

$\phi_t(x)$ as $t \to \infty$.

- Are there closed orbits ($\phi_T(x) = x$ for some $T \neq 0$ & x)
  How many [illegible]
- Are there inv. measures [illegible] $\phi_t^* \Omega = \Omega$
  more [illegible] invariles ~~and~~ the same

3 PARTS:

PART 1: Vector field dynamics:

1. M: compact oriented 3 mfld

α = contact 1 form on M so $\alpha \wedge d\alpha > 0$

v = Reeb vector field, $v \lrcorner d\alpha = 0$ $v \lrcorner \alpha = 1$.

- Of interest are closed orbits of v.

Thm (W...): Any such v has a closed orbit.

- What if v is not of "contact type"?

Stable Hamiltonian system

a, f, ω — a is 1-form, f is function, ω is 2 form

$da = f\omega$ — $a \wedge \omega > 0$

So f can be zero, pos, neg. — $v \lrcorner \omega = 0$, $v \lrcorner a = 1$

Thm (Mike Hutchings & ...): v has a closed orbit except ~~...~~ maybe if $M = T^3$

$$\begin{array}{c} M = T^3 \\ \downarrow \\ S^1 \end{array}$$

Does [illegible] [illegible] general [illegible]?

- Some [illegible] ∃ volume preserving v-
  Harrison, [illegible] fields with no
  Kuperberg [illegible] closed orbits

- If v has no closed orbits what are [illegible]
  [illegible]

- Minimal dynamical system: Every trajectory is dense?
  Gottschalk conj: ∃ none on S^3 ??

- If v is volume preserving then $\cancel{\text{[illegible]}}$ $\mathcal{L}_v \Omega = 0$ so Ω gives invariant measure. A vector field is said to be "uniquely ergodic" if Ω is only invariant measure [illegible] [illegible] [illegible] [illegible] [illegible] of diffeomorphisms.

What can be said about [illegible] [illegible] [illegible] [illegible]?

- Only a few are known to exist:

  - e.g. Unit tangent bundle of surface of genus $g > 1$: Horocycle flow is uniquely ergodic.

Say that volume preserving v.f. is exact if $v \lrcorner \Omega = db$
(e.g. horocycle flow is exact.)
Define $\mathcal{H}_v = \int_M b \wedge (v \lrcorner \Omega)$ (Asymptotic winding #)
This does not depend on [illegible] anti-deriv. b

- Thm (C.T): If v is uniquely ergodic then $\mathcal{H}_v = 0$.

× [illegible] v.f. are [illegible]

× Any flow on a $\mathbb{Q}$-homology S^3 is exact.

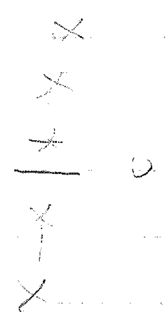
× Horocycle flow is exact & in fact
$v \lrcorner \Omega = db$ & $b \wedge db = 0$
So v is tangent to a foliation
(leaves are gen. by v & geodesic flow.)

PART 2: Vector field

- Spectral dynamics:

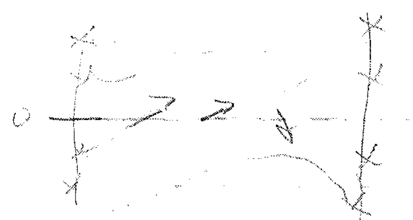
$\mathcal{H}$ = Hilbert space.
L: self-adj. operator on some dense domain [illegible]

L has discrete spectrum in $\mathbb{R}$ w/ no accum. pt. & finite multipl.



- $\{L_t\}_{t\in[0,1]}$ one family of such op. with $0 \notin \mathrm{Spec}(L_0),\ \mathrm{Spec}(L_1)$

Spectral flow:



[illegible]

- Eg: $L_r = \left[i\frac{\partial}{\partial\theta} + r \right]$ on $L^2(S^1; \mathbb{C})$
$S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Then $L_r e^{in\theta} = -n + r$

$SF = \ldots \sim r + O(1)$

Exact Morse on M^3. Put Riem. metric with vol. form $= \Omega$

$$\begin{array}{ccc} \tilde{F} & \supset & U(2) \\ \downarrow & & \downarrow \\ Fr & = & SO(3) \\ \downarrow & & \\ M & & \end{array}$$

Spin$_c$ struct $\Longleftrightarrow H^2(M;\mathbb{Z})$

$\tilde{F} \times_{U(2)} \mathbb{C}^2 = \mathbb{S}$ spinor bundle

Module for Cliff mult

$T^*M \xrightarrow{cl} \mathrm{End}(\mathbb{S})$

$cl(a)^* = -cl(a)$

$cl(a)cl(b) + cl(b)cl(a) = -2\langle a,b\rangle$

$\det \mathbb{S} = \tilde{F} \times_{\det} \mathbb{C}$ $\quad (= \Lambda^2 \mathbb{S})$

Conn A on $\det \mathbb{S}$ $\Rightarrow$ Cov. deriv on $C^\infty(M;\mathbb{S})$

$D_A = cl(\nabla_A) =$ S. adj on $\Gamma(M;\mathbb{S})$

Fix $\Gamma =$ conn on $\det \mathbb{S}$, $\quad \theta =$ 1-form on M

$r \to D_{\Gamma + ir\theta/2}$ $\quad$ look at spectral flow

e.g. $d\theta = \omega_v = v \lrcorner \Omega$

Note: if $c_1(\det \mathbb{S}) =$ torsion, then SF does not dep. on choice of conn.

Thm: $sf_{[D_r]} = \frac{1}{4\pi^2} r^2 \int b \wedge db_v + O(r^{3/2})$

(assumption: see # 1).

[illegible]

- [illegible]

ex. for [illegible] sf ~ r^2 vol.

Part 3: Non-linear dynamics (SW) of vector field.

Seiberg-W equ.

A = conn on det S
ψ = sect. of S

$$*F_A = r(\psi^\dagger\tau\psi - ia)$$
$$D_A\psi = 0$$

$\langle b, \psi^\dagger\tau\psi\rangle \equiv \psi^\dagger\, cl(b)\, \psi$

Interest is in how solutions vary as $r \to \infty$.

(Are there any? of what sort? What happens as $r \to \infty$?)

$v \lrcorner d\omega = \omega_v \ldots$ Choose a [illegible]

Choose $\alpha = \ast\omega_v$

Let $E(A) = i\int a \wedge F_A$

Relation to Conf. dynamics:

Thm: Suppose $\exists \{(A_n, \Psi_n), r_n\}$ [illegible]

$r_n \to \infty$, (A_n, Ψ_n) obey SW eq.

and

- $E(A_n) \leq \mathcal{E} < \infty$
- $\sup_M 1 - |\psi_n| > \delta > 0$.

Then $\exists$ at least one closed orbit of v.

- If $E(A_n) \to \infty$, then $f \to \frac{r}{E(A_n)}\int f(1-|\psi_n|^2)$ conv. to an inv. measure. This is not so if $\Delta_v \neq 0$.

Floer Homology:

- SW eq. are variational eqs. of funcl.

$$\mathfrak{a} = \frac{1}{2}(\operatorname{cs}(A) - r\hat{E}) + 2r\int_M \psi^\dagger D_A \psi$$

Focus on case $c_1(\det \mathbb{S}) =$ torsion.

$\Gamma =$ flat connection on $\det$ [illegible]

$A = \Gamma + \hat{a}_A$

$$\mathrm{cs} = \int \hat{a}_A \wedge d\hat{a}_A \qquad \hat{E} = \int b \wedge F_A$$

$db = \omega$

- Critical pts of $\mathfrak{a}$ $\Leftrightarrow$ solutions of SW

- Floer hom = $\mathbb{Z}$ module generated by [illegible] in part by equiv. classes of solutions

$$A,\Psi \to A - 2u^{-1}du, \quad \Psi \to u\Psi \qquad u : M \to S^1$$

Irred. $\Rightarrow$ $|\Psi| \neq 0$ [illegible]

Red. [illegible] $\Psi = 0$ [illegible] stabilizer S^1.

Defined in equiv. cohomology [illegible] and as H^* [illegible]

$\mathbb{Z}$-graded

- Thm (Kron-Mr): $\exists$ [illegible] s.t. $SW^{*} \neq 0$.

Digression about Floer homology

What is point? It guarantees solutions to an equation!! If homol $\neq 0$, there must be some solutions.

(Every fn. on compact mfld has at least $\sum_i b_i$ crit. pts.)

[illegible] Floer hom. to talk about [illegible]

[illegible] non [illegible] [illegible] generators
∃ solutions.

- KM's Thm on the face of it isn't quite enough. ∃ ∞ non zero [illegible] contributions [illegible]

$$*F_A = -ir*\omega \qquad A = \Gamma + ib$$
$$\Delta b = [illegible]$$

$$E = i\int *\omega \wedge F_A = r\int *\omega\wedge\omega = r\cdot \mathrm{Vol}$$

- When [illegible] case a = contact,
  $\mathfrak{e}_v = \int \omega\wedge b \neq 0$
  Some other cases?

⇒ Spectral flow dynamics:
if $\mathfrak{e}_F \neq 0$ then ∃ irreducible solutions for SW.

- Maximum principle finds $E = r(1-|\psi|^2) = O(1)$

- $\mathfrak{a} = \frac{1}{2}(a - r\hat{E}) + \psi^\dagger D_A \psi \qquad \hat{E} =$

$|\hat{E}/E| \geq \delta > 0 \Rightarrow$ ∃ closed orbit

$|\hat{E}/E| \to 0$ ∃ measures $\qquad \hat{E} = \dfrac{\int q(1-|\psi|^2)}{\int(1-|\psi|^2)}$

Thus,

$$\mu_n(\Omega) = \frac{\int f(1-|\varphi_n|^2)}{\int (1-|\varphi_n|^2)} \quad \text{has} \quad \mu_n(\Omega) = 1$$

$$\mu_n(\ldots) \to 0.$$

But $\int_{\Omega} \ldots = \Delta_v \neq 0.$

- These conditions are proved in [illegible] with [illegible].

- Long story.