An introduction to homogeneous dynamics

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$$x^2 - dy^2 = 1$$

Fact: If $d \neq$square, $\exists\ (x,y) \in \mathbb{Z}^2$, not $(\pm 1, 0)$

D_t: stretches by e^t along one axis
stretches by e^{-t} along the other axis
$D_t \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

As t varies, $D_t(p)$ moves along a level set. We'll show $\exists\ t > 0$ s.t. $D_t(1,0) \in \mathbb{Z}^2$
$x^2 - dy^2 =$const.

$$D_t \mathbb{Z}^2 = \mathbb{Z}^2$$

A lattice $L \subset \mathbb{R}^2$ is a grid containing $(0,0)$

$$L \colon = \{mv_1 + nv_2 \colon m, n \in \mathbb{Z}\}$$

area$(\mathbb{R}/L)=$ det $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\mathcal{L} = \{\text{lattices } L \subsetneq \mathbb{R}^2 \text{ s.t. area}(\mathbb{R}/L) = 1\}$$

$$[Z^2] \in \mathcal{L}, \quad D_t \colon \mathcal{L} \to \mathcal{L}$$

Want $D_{t_0}[\mathbb{Z}^2]$, $t_0 > 0$

Fact: $\mathcal{L}$ is $\boxed{\text{almost}}$ compact $\Rightarrow \exists\ t_1, t_2$ s.t. $D_t[\mathbb{Z}^2]$, $D_{t^2}[\mathbb{Z}^2]$ are "close".

Fact: Discreteness: If $D_{t_1}[\mathbb{Z}^2]$, $D_{t_2}[\mathbb{Z}^2]$ are close enough then there is

$$t_2' \sim_{\text{close}} t_2$$

so that $D_{t_1}[\mathbb{Z}^2] = D_{t_2'}[\mathbb{Z}^2]$.

Reason: For any t, $D_t\ \mathbb{Z}^2 \subset \bigcup_{n \in \mathbb{Z}} (x^2 - dy^2 = n)$

So t_1, t_2 so that $D_{t_1}[\mathbb{Z}^2] = D_{t_2'}[\mathbb{Z}^2]$

$$D_{t_1 - t_2'}[\mathbb{Z}^2] = [\mathbb{Z}^2]$$

Key property of $f(x,y) = x^2 - dy^2$
Many linear automorphisms

$$g \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$f(g(xy)) = f(x,y).$$

(This is true for all quadratic forms + other examples)

Exercise: Solve $x^2 - dy^2 = 2$ by some variant.

Space of Lattices:

$$\mathcal{L}_n = \{\text{lattices } L \subseteq \mathbb{R}^n \text{ with vol}(\mathbb{R}^n/L) = 1\}$$

$G = SL(n,\mathbb{R})$ acts transitively on $\mathcal{L}_n$; the map $g \mapsto g[\mathbb{Z}^n] \in \mathcal{L}_n$ identifies $\mathcal{L}_n$ with

$$SL(n,\mathbb{R})/ \underbrace{SL(n,\mathbb{Z})}_{\text{stabilizer of } \mathbb{Z}^n}$$

<u>Properties:</u>

(1) $\boxed{\text{Almost}}$ compact: Fix open nbhd U of 0 in $\mathbb{R}^n$, then $\{L \in \mathcal{L}_n \colon L \cap U = \{0\}\}$ is compact (Mahler's criterion).
(2) $\mathcal{L}_n$ has a $SL_n\mathbb{R}$-invariant probability measure.

<u>Exercise:</u> How would you produce random lattices?

Our main concern is dynamics of close subgroups $H \subset SL(n,\mathbb{R})$ acting on $\mathcal{L}_n$. (e.g. $H = \{D_t\}_{t\in\mathbb{R}} \subset SL_2(\mathbb{R})$)

Theorem 0.1 (Howe-Moore). *Any non-compact $H \subset SL_n(\mathbb{R})$ acts ergodically on $\mathcal{L}_n$. (mixingly)*

<u>Example:</u>

$$H = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\} \subset SL(2,\mathbb{R})$$

H acts ergodically on $SL(2,\mathbb{R})/\Omega(2,\mathbb{Z})$, $h \cdot gSL(2,\mathbb{Z}) \mapsto hg \;\; SL(2,\mathbb{Z})$

$$\Leftrightarrow$$

$SL(2,\mathbb{Z})$ acts ergodically on

$$\underbrace{SL(2,\mathbb{Z})/H}_{\mathbb{R}^2-\{0\}}$$

<u>Lattice-point counting:</u> Given poly $f \in \mathbb{Z}[x_1,\dots,x_k]$

$$\{\overline{x} \in \mathbb{Z}^n \colon f(x) = 0\}$$

Study the growth of

$$N(t) = \{x \in \mathbb{Z}^n, \|x\| < T \colon f = 0\}$$

When f has many linear automorphisms one can hope to analyze $N(T)$ via dynamics on $\mathcal{L}_n$.

$$\boxed{\text{“Markov triples”} \quad x^2 + y^2 + z^2 + xyz = 0 \quad (x,y,z) \mapsto (x,-y,z-xy)}$$

<u>Example:</u> $x_1x_2 - x_3x_4 = 1$

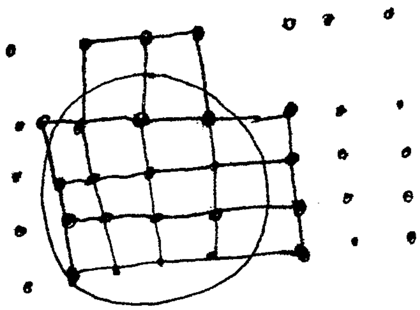
$$\det \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

$$N(T) = \#\{\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \in SL_2\mathbb{Z},\ \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} < T\}$$

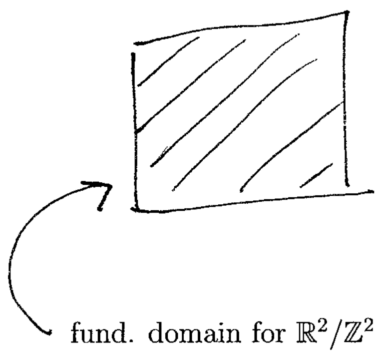
We'll think of it as a nonabelian analogue of

$$N'(T) = \#\{(x,y) \in \mathbb{Z}^2 \colon \sqrt{x^2+y^2} \le T\}$$

$$N'(T) \sim \pi T^2 + \text{error}$$



$$\mathrm{Ball}_{T-2} \subset \text{Thing}$$
$$\subset \text{Ball}$$
$$\mathbb{Z}^2 \subset \mathbb{R}^2$$
$$SL_2\mathbb{Z} \subset SL_2\mathbb{R}$$



fund. domain for $\mathbb{R}^2/\mathbb{Z}^2$

$$\mathrm{Ball}_T = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a^2+b^2+c^2+d^2 \leq T^2\}$$

$$\bigcap_{g\in\Omega} g\mathrm{Ball}_T \subseteq \text{Thing} \subseteq (\text{Ball})\cdot\Omega$$

$$\Omega \text{ fixed, compact}$$

$$\frac{vol(\mathrm{Ball}_t\cdot\Omega)}{vol(\mathrm{Ball}_t)} \nrightarrow 1$$

What you need to get $o(T)$ in $\mathbb{Z}^2$-problem:

the projection of circle of radius R to $\mathbb{R}^2/\mathbb{Z}^2$ is uniformly distributed

Similarly, the statement

(∗) | the projection of $\delta\mathrm{Ball}_T$ to $SL_2\mathbb{R}/SL_2\mathbb{Z}$ is uniformly distributed (w.r.t. $SL_2(\mathbb{R})$-inv. prob.) |

$$\Leftarrow$$

$$\lim_{T\to\infty}\frac{N(T)}{vol(\mathrm{Ball}_T)} \to 1$$

(∗) would follow (by an approximation argument). Given any $x \in SL_2\mathbb{R}/SL_2\mathbb{Z}$, the orbit of x under $H = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}$ is uniformly distributed

i.e. for $f \in C_c(\Omega_2\mathbb{R}/SL_2\mathbb{Z})$

$$\frac{1}{T}\int_0^T (f\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x) \to \int_{SL_2\mathbb{R}/SL_2\mathbb{Z}} f$$

We need to understand the distribution of H-trajectories ($\Rightarrow$ Ratner's Theorem)
Completely answered by

$$\begin{array}{ccc} H & & SL_n\mathbb{R} \\ & \diagdown & \downarrow \\ & & G \end{array}$$

$$\begin{array}{c} SL_n\mathbb{Z} \\ \downarrow \\ \Gamma \end{array}$$