Harmonic Analysis and Additive Combinatorics
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22 August 2008

Key words: harmonic analysis, discrete Fourier transform, arithmetic progressions, Freiman's theorem, Roth's theorem, Bohr sets, restriction estimates.

Theorem 0.1 (Freiman). *$A \subseteq \mathbb{Z}$, $|A + A| \le K\,|A|$. Then A is contained in a generalized arithmetic progression P such that $|P| \le C\,|A|$, $\dim P \le d$.*
C,d depend only on K. (Can take $d = O(K^2)$, $c = e^{O(K^2)}$.)

Lemma 0.2 (Bogolyubov). *$A \subset \mathbb{Z}_N$, $|Z| = \delta N$, let $\Gamma = \{\xi \in \mathbb{Z}_N\colon \left|\widehat{A}(\xi)\right| \ge \frac{1}{2}\delta^{3/2}\}$. Then $2A - 2A = \{a + b - c - d\colon a, b, c, d \in \Lambda\}$ contains the Bohr set:*

$$B = B(\Gamma, \frac{1}{10}) = \{x \in \mathbb{Z}_N\colon \left\|\frac{x\xi}{N}\right\| < \frac{1}{10} \text{ for all } \xi \in \Gamma\}.$$

Why is this useful?
-B contains a large GAP (generalized AP) P_0
-Use P_0 to construct P

Proof of Lemma: $x \in 2A - 2A \iff$

$$0 < \#\{a, b, c, d \in A\colon a + b - c - d = x\}$$
$$= \sum_{a,b,c,d}\left[A(a)A(b)A(c)A(d) - \frac{1}{N}\sum_{\xi\in\mathbb{Z}_N} e^{-2\pi i(a+b-c-d)\xi/N}\right]$$
$$= N^3 \sum_{\xi\in\mathbb{Z}_N}\left|\widehat{A}(\xi)\right|^4 e^{-2\pi i x\xi/N}$$
$$= N^3 \sum_{\xi\in\mathbb{Z}_N}\left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr)$$

Want to prove:
for all $x \in B$

$$0 < N^3 \sum_{\xi\in\mathbb{Z}_N}\left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr)$$
$$= \left|\widehat{A}(0)\right|^4 + \sum_{\xi\in\Gamma,\xi\neq 0}\left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr) + \sum_{\xi\notin\Gamma}\left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr)$$

$$\left|\widehat{A}(0)\right|^4 = \delta^4 \text{ (last time)}$$

$$\sum_{\xi\in\Gamma,\xi\neq 0}\left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr) > 0$$

because $2\pi x\xi/N \le \frac{2\pi}{10}$ for $x \in B$, $\xi \in \Gamma$ and $\cos(2\pi x\xi/N) > 0$.

$$\left|\sum_{\xi\notin\Gamma}\left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr)\right| \le \max_{\xi\notin\Gamma}\left|\widehat{A}(\xi)\right|^2 \cdot \sum_{\xi}\left|\widehat{A}(\xi)\right|^2 \le \frac{\delta^3}{4}\cdot\delta$$

Thus,

$$\left|\widehat{A}(0)\right|^4 + \sum_{\xi\in\Gamma,\xi\neq 0} \left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr) + \sum_{\xi\notin\Gamma} \left|\widehat{A}(\xi)\right|^4 \cos\bigl(2\pi x\xi/N\bigr) \geq \delta^4 - \frac{\delta^4}{4} > 0.$$

□

Roth's Theorem in Sparse Sets: Assume $A \subset \mathbb{Z}_N$, $|A| \cong N^{1-\theta}$, $\theta > 0$.

Must A contain 3-AP's?
-In general, no.
-But: Suppose $A \subset W$, $|W| \cong N^{1-\theta}$, $\left|\widehat{W}(\theta)\right| \leq N^{-10\theta}$, θ close to zero. Then A contains 3-AP's if N is large enough.

Kohayakawa-Łuczak-Rödl, Green, Green-Tao, Tao-Vu

Proof (sketch): Let $f(x) = N^\theta A(x)$. Then

$$\sum_x f(x) = N^\theta |A| = N^\theta \delta N^{1-\theta} = \delta N.$$

Transference Principle: $f = f_1 + f_2$, $0 \leq f_1 \leq K$ (indep. of N)
f_2 "random" $\left|\widehat{f_2}(\xi)\right| \leq \epsilon$, $\epsilon = \epsilon(\delta)$.

Define: $f_1, f_2 \colon \Gamma = \{\xi \in \mathbb{Z}_N \colon \left|\widehat{f}(\xi)\right| > \epsilon\}$

$$B = B(\Gamma, \epsilon) = \{x \in \mathbb{Z}_N \colon \left\|\frac{x\xi}{N}\right\| < \epsilon \text{ for } \xi \in \Gamma\}$$

$$f_1(x) = \frac{1}{|B|^2} \sum_{y_1,y_2\in B} f(x + y_1 - y_2), \quad f_2(x) = f - f_2$$

$$\frac{1}{N^2}\sum_{x,r} f(x)f(x+r)f(x+2r) = \sum_\xi \widehat{f}(\xi)^2 \widehat{f}(-2\xi)$$
$$= \sum_\xi \widehat{f}_1(\xi)^2 \widehat{f}_1(-2\xi) + \sum_\xi \widehat{f}_2(\xi)^2 \widehat{f}_2(-2\xi) + \dots$$

The first term is bounded by below (Roth-Varnavides)
We have $|\widehat{f_2}(\xi)|$ small. We want to use it to prove that the remaining terms above are small.

□

Problem:

$$\sum_\xi \left|\widehat{f}(\xi)\right|^2 = \frac{1}{N}\sum |f(\xi)|^2$$
$$= \frac{1}{N}\sum_{x\in A} N^{2\theta} = \frac{1}{N}\delta N^{1-\theta} N^{2\theta} = \delta N^\theta$$

which is BAD.

Restriction Estimates: Work in $\mathbb{R}^d$, σ=Lebesgue measure on S^{d-1}

$$f \colon S^{d-1} \to \mathbb{C}$$

$$\widehat{f\ d\sigma}(\xi) = \int f(x)e^{2\pi i\xi x}\ d\sigma(x)$$

Is $f \mapsto \widehat{f\ d\sigma}$ bounded $L^p(S^{d-1}) \to L^q(\mathbb{R}^d)$?

Trivial: $L^1 \to L^\infty$ bdd.

Stein-Tomas: $L^2 \to L^q$, $q \geq \frac{2d+2}{d-1}$

Conjecture (Stein): $L^\infty \to L^q$, $q > \frac{2d}{d-1}$

Number-theoretic analogues: Bourgain, Green, Tao-Vu. Work in $\mathbb{Z}_N$, with σ replaced by

$$\frac{1}{|W|}\sum_{x\in W}\delta_x$$

then for f as above, $\left\|\widehat{f}\right\|_q \leq M$ independently of N for some $2 < q < 3$.
We use this to estimate the error terms, e.g.

$$\left|\sum_{\xi}\widehat{f}_2(\xi)^2\widehat{f}_2(-2\xi)\right| \leq \max_{\xi}\left|\widehat{f}_2(\xi)\right|^{3-q}\cdot\|\widehat{f}_2\|_q^q \leq M^q \max_{\xi}\left|\widehat{f}_2(\xi)\right|^{3-q}$$