An introduction to homogeneous dynamics

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Unipotent subgroups: A unipotent one-parameter subgroup of $SL(n,\mathbb{R})$ is one of the form

$$\{\exp(tX) \colon t \in \mathbb{R}\}$$

(X is a $n \times n$ matrix with trace$X = 0$)
where X is nilpotent (equiv.: $X^n = 0 \Leftrightarrow$ all gen. evals 0)
(e.g. $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$)
In this case $\exp(tX)$ is polynomial in t.

Example: $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

Nonexample: $D_t = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$ (not unipotent)

Set

$$N = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}$$

$$A = \{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\}$$

Exercise: For $g \in SL_2\mathbb{R}$, consider gAg^{-1}. Show as $gA \in SL_2\mathbb{R}/A \to \infty$, and "limit" of these subgroups is one-parameter unipotent

Yesterday:

$$\mathcal{L}_n = \text{(lattices of volume 1 in } \mathbb{R})$$

$G \overset{\text{closed}}{\subset} SL_n\mathbb{R}$, let $\Gamma \subset G$ be a discrete subgroup such that G/Γ has G-invt. prob. measure.

Let $U(t) = \exp(tX)$ be unipotent 1-parameter subgroup of G.

Examples:

(1) $G = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}$ with det 1,

$$\Gamma = G \cap SL(3,\mathbb{Z})$$

$$G/\Gamma = \text{(affine lattices in } \mathbb{R}^2 \text{ of area 1)}$$

$$U(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) $G = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ $\Gamma = G \cap SL_3(\mathbb{Z})$

G/Γ example of nilmanifold.

Theorem: (UD) $x \in G/\Gamma$. Then there exists a homogeneous measure ν on G/Γ s.t.

$$\frac{1}{T}\int_0^T f(U(t))\,dt \to \nu(f),\ T \to \infty$$

Homogeneous: "The natural measure on sub-homogeneous space"
i.e. $\exists$ closed $H \subset G$, H contains $\{U(t)\}$, such that Hx is closed and ν is the H-invt. prob. measure on Hx.

eg: G/Γ $\mathcal{L}_2$, $U(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ –either $U(t)x$ is periodic, or $U(t)x$ becomes u.d. on G/Γ

eg 2: $G = SL_2\mathbb{R} \times SL_2\mathbb{R} \supset \Gamma = SL_2\mathbb{Z} \times SL_2\mathbb{Z}$

$$G/\Gamma = \mathcal{L}_2 \times \mathcal{L}_2$$

Here there is another possibility, $\exists$ points such that $U(t)x$ becomes uniformly distributed in an embedded $SL_2\mathbb{R}/\Gamma'$

eg. $x = (L, L')$, L, L' commenurable.

Corollary: $\overline{U(t)x}$ is homogeneous (i.e. of the form Hx, H is closed)

Theorem: (Measure Classification) Any $U(t)$-invariant, $U(t)$-ergodic probability measure on G/Γ is homogeneous (as in UD).

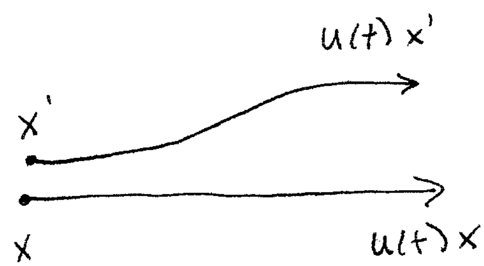
Remark: The statements of UD, MC are false for $SL_2\mathbb{R}/SL_2\mathbb{Z}$ if you replace $U(t)$ by A (exercise).

Exercise: $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $x \in SL_2\mathbb{R}/SL_2\mathbb{Z}$ is not $U(t)$-periodic, show that $\{x, Ux, U^2x, U^3x, \ldots\}$ is u.d.

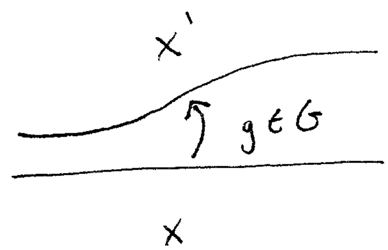
One idea in the proof of MC $U(t)$ unipotent one-parameter, μ invariant ergodic measure. Show that μ is invariant by more and more elements in G.

By ergodic theorem, μ-almost all $x \in G/\Gamma$ are generic ($f \in C(X)$ compact support, $\frac{1}{T}\int f(U_t x) \to \int f$).

Find two nearby generic pts x, x'



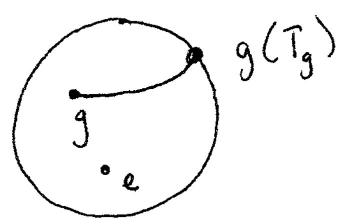
We can find long segments of the trajectories which are "approximately parallel".



$\Rightarrow g \cdot \mu \sim \mu$

1.) Fix ball $B \subseteq G$ containing identity. Fix right inv. metric on G. For any $g \in G$, put $g(t) = U(t)gU(-t) = U(t)gU(t)^{-1}$. As long as g doesn't commute with $U(t)$, $g(t)$ will exit B.

$$Tg = \inf\{t\colon g(t) \notin B\}$$
$$g^* = g(T_g)$$



"approximate parallel"
$\forall\, \epsilon > 0$, $\exists\, \delta$ such that for $t \in [(1-\delta)Tg, Tg] \Rightarrow d(g(t), g^*) < \epsilon$. (because $g(t)$ is polynomial of bounded degree).

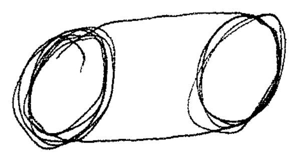
Lemma Suppose x_n, x_n' are generic points so that $x_n' = g_n x_n$ and $g_n \to id$. Let g^* =any limit of g_n^*. Then μ is invt. under g^*.

Proof: $f \in C_c(G/\Gamma)$. Take $\epsilon > 0$, δ as above

$$\begin{aligned}
\mu(f) &\overset{x_n' \text{ generic}}{\sim} \frac{1}{\delta T g_n} \int_{(1-\delta)Tg_n}^{Tg_n} f(U(t)x_n') \\
&= \frac{1}{\delta T g_n} \int_{(1-\delta)Tg_n}^{Tg_n} f(g_n(t)U(t)x_n) \\
&\overset{\text{by } \epsilon-\delta \text{ thing}}{\sim} \frac{1}{\delta T g_n} \int f(g_n^*(t)U(t)x_n) \\
&\overset{x_n \text{ generic}}{\sim} \mu(g_n^* f)
\end{aligned}$$

$g_n^* f(x) = f(g_n^* x)$
Pass to the limit:
$\mu(f) = \mu(g^* f) \Rightarrow \mu$ is g^*-invt.

MC $\Rightarrow$ UD
What we don't want:



Again, the fact that $t \mapsto g(t)$ is polynomial is critical to rule it out.

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