On the sumsets of infinite sequences

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Consider a set $A = \{a_1 < a_2 < \ldots < a_n < \ldots\}$. Then $S_A\colon = \left\{\sum_{x \in B} x\colon B \subset A,\ |B| = \text{finite}\right\}$, $lA\colon = \{a_{i_1} + a_{i_2} + \ldots + a_{i_l}\}$, and $l^*A\colon = \left\{\sum_{a \in B} a\colon B \subset A,\ |B| \le l\right\}$.

Suppose $A \subset \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

$|A + A| \ge \max\{p, |A| + |A| - 1\}$

Define $A(n)$ to be the number of elements of A not exceeding n. If A is the set of squares, then $A(n) \le \sqrt{n}$.

We say that A is *complete* if S_A contains all large enough numbers. A is *subcomplete* if S_A contains an infinite arithmetic progression.

Conjecture 0.1. ***Erdős (1962):*** *Assume A satisfies:*

(1) $A(n) > C\sqrt{n}$

(2) *the intersection of S_A with any infinite arithmetic progression is nonempty.*

Then A is complete.

Conjecture 0.2. ***Folkman:*** *Assume A satisfies: $A(n) > C\sqrt{n}$. Then A is subcomplete.*

Conjecture 0.3. ***Erdős:*** *Assume A satisfies: $A(n) > n^{\frac{\sqrt{5}-1}{2}}$. Then A is subcomplete.*

Conjecture 0.4. ***Folkman:*** *Assume A satisfies: $A(n) > n^{\frac{1}{2}+\epsilon}$. Then A is subcomplete.*

Theorem 0.5. ***Hegyvári, Luczak, Shoen:*** *If $A(n) \ge n^{\frac{1}{2}}\sqrt{\log n}$ then A is subcomplete.*

Theorem 0.6. ***Van H. Vu:*** *If $A(n) \ge c \cdot n^{\frac{1}{2}}$ then A is subcomplete.*

Conjecture 0.7. *If $A \subseteq [0, n]$ and $|A| \ge 2\sqrt{n}$ then S_A contains arithmetic progressions of length n.*

Theorem 0.8. *(Bourgain): Suppose $A \subset [0, n]$, and $|A| \ge \gamma n$. Then $A + A$ contains an arithmetic progression of length $e^{\sqrt{\log n}}$.*

Theorem 0.9. *(Green): Suppose $A \subset [0, n]$, and $|A| \ge \gamma n$. Then $A + A$ contains an arithmetic progression of length $e^{\log n^{\frac{2}{3}}}$.*

Theorem 0.10. *(Halberstam, Ruzsa, Freiman): Suppose $A \subset [0, n]$ and $|A| = \gamma n$ with $\gamma < \frac{1}{\log n}$. Then $|A + A + A| \ge n^{\frac{\gamma}{6}}$.*

Theorem 0.11. *(Ruzsa): There exists some m, and some $A \subset [0, n]$, so that $|A| \ge \gamma n$, but $A + A$ does not contain an arithmetic progression of length $e^{\log n^{1-\epsilon}}$. Here ϵ is dependent on γ.*

Theorem 0.12. *(G. Freiman, A. Sarkozy): $l\,|A| \ge kn \Rightarrow lA$ contains arithmetic progressions $\ge |l|\,|A|$.*

l^*A, $|l|\,|A| \ge \log n \cdot n \Rightarrow lA$ contains arithmetic progressions $\ge |l|\,|A|$.

Theorem 0.13. ***(Van Vu, Szemerédi)***
$l\,|A| \ge kn \Rightarrow l^*\,|A|$ contains arithmetic progressions $\ge n$.