Nilsystems in ergodic Theory I'

A digression: Nilsystems and polynomials

Polynomial sequences in nilsystems.
Let $X = G/\Gamma$ be a nilmanifold. A polynomial sequences in X is a sequence of the form

$$x_n = g_1^{p_1(n)} g_2^{p_2(n)} \dots g_k^{p_k(n)} \cdot x$$

where $g_1, g_2, \dots, g_k \in G$,
$p_1, p_2, \dots, p_k$ are integer polynomials and $x \in X$.

We have already showed how a polynomial sequence in $\mathbb{T}$ can be produced by a nilssytem.
The next theorem says that any polynomial sequence in a nilsystem can be produced by some nilsystem:

Nilsystems can be considered as the "ecological niche" of polynomials.

Theorem (Leibman)**.** *Let $X = G/\Gamma$ be a nilmanifold and let $\{x_n \colon n \in \mathbb{Z}\}$ be a polynomial orbit in X.*

Then there exist a nilsystem (Y, S), a smooth map $\pi \colon Y \to X$ and a point $y \in Y$ such that

$$x_n = \pi(T^n y) \text{ for every } n \in \mathbb{Z}\ .$$

Corollary (Leibman)**.** *Let (X, T) be a nilsystem.*
If $p_1, \dots, p_k$ are integer polynomials and $f_1, \dots, f_k$ are continuous functions on X then the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{p_1(n)} x) \dots f_k(T^{p_k(n)} x)$$

converge everywhere.

The distribution of the finite sequence $\{n\alpha \bmod 1 \colon 0 \leq n < N\}$ is well understood.
In a recent paper, Green & Tao study the distribution of finite blocks of polynomial orbits in a nilsystem.

Nilsystems in ergodic Theory II

Multiple recurrence and convergence,
Seminorms on $L^\infty(\mu)$
and the structure theorem

Henceforth, (X, μ, T) is an ergodic measure preserving system.

Notation: Functions on X are assumed to be real valued.
When f is a function on X, $T^n f = f \circ T^n$.

1. Some results about multiple recurrence and recurrence.

Nilsystems play a central role in the proof of some (relatively) recent results about multiple convergence: they are characteristic factors.

Examples:

Let $f_1, \dots, f_k$ be bounded functions on X.

[**H & Kra**] The multiple ergodic averages

$$\frac{1}{N} \sum_{n<N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k$$

converge in $L^2(\mu)$.

[**H & Kra, Leibman**] If $p_1, p_2, \dots, p_k$ are integer polynomials then the multiple ergodic polynomial averages

$$\frac{1}{N} \sum_{n<N} T^{p_1(n)} f_1 \cdot T^{p_2(n)} f_2 \cdot \ldots \cdot T^{p_k(n)} f_k$$

converge in $L^2(\mu)$.

It should be possible to study also averages like

$$\frac{1}{\pi(N)} \sum_{p<N} T^p f_1 \cdot T^{2p} f_2 \cdot \ldots \cdot T^{kp} f_k$$

(the average is for p prime $< N$)
but this would need deep results of number theory.

Questions on multiple recurrence

We recall:

Let $A \subset X$ be a set of positive measure.

[**Furstenberg**] There exists $n > 0$ such that

$$\mu\big(A \cap T^{-n} A \cap T^{-2n} A \cap \cdots \cap T^{-kn} A\big) > 0\ .$$

[**Bergelson & Leibman**] Let $p_1, p_2, \ldots, p_k$ be integer polynomials with $p_i(0) = 0$.
Then there exists $n > 0$ such that

$$\mu\big(A \cap T^{-p_1(n)}A \cap T^{-p_2(n)}A \cap \cdots \cap T^{-p_k(n)}A\big) > 0 \ .$$

Several recent papers deal with more precise versions of these results.

Question. Can we say something about

$$\sup_{n>0} \ \mu\big(A \cap T^{-p_1(n)}A \cap T^{-p_2(n)}A \cap \cdots \cap T^{-p_k(n)}A\big)$$

under some additional conditions about the system and/or the polynomials?
(As above, each p_i is an integer polynomial and $p_i(0) = 0$)

[**Bergelson**] If X is weakly mixing then this sup is $\geq \mu(A)^{k+1}$.

More recent result:
[**Frantzikinakis & Kra**] Same conclusion if X is totally ergodic and the polynomials are linearly independent.

The important point is to give the best possible definition for the complexity of the family $\{p_1, \ldots, p_k\}$ of polynomials.

Furstenberg's Theorem implies Szemerédi's Theorem and Bergelson Leibman's Theorem implies the
Polynomial Szemerédi's Theorem (Bergelson & leibman) Let $p_1, p_2, \ldots, p_k$ be integer polynomials with $p_i(0) = 0$ and $E \subset \mathbb{Z}$ be a set of positive density. Then there exist $n > 0$ and $a \in \mathbb{Z}$ such that

$$\big\{a, a + p_1(n), +a + p_2(n) + \ldots, a + p_k(n)\big\} \subset E \ .$$

Question. Can these results be improved under some conditions on the complexity of the family of polynomials and/or the uniformity of the set E?

Similar questions arise in the finite setting: How many configurations of some given type occur in a subset of $[1, N]$ of given density?
More difficult: How many configurations of some given type occur in the primes $\leq N$?
The guess is that the methods developed for the ergodic case can give ideas for the finite setting.

Back to ergodic theory...

2. Seminorms on $L^\infty(\mu)$

We recall that (X, μ, T) is an ergodic system.

For $f \in L^\infty(\mu)$, we define by induction:

$$|||f|||_1 = \left|\int f \, d\mu\right|$$

and for $k \geq 1$,

$$|||f|||_{k+1} = \Big(\lim_{N\to+\infty} \frac{1}{N} \sum_{n<N} |||f \cdot T^n f|||_k\Big)^{1/2^{k+1}} .$$

We do not prove here that the limits exist and that these expressions define seminorms on $L^\infty(\mu)$.

These seminorms are similar to the Gowers norms.

Using the seminorms.

Theorem. *Let $f_1, \dots, f_k \in L^\infty(\mu)$ satisfy $\|f_i\|_\infty \leq 1$. Then*

$$\limsup_{N\to+\infty} \Big\| \frac{1}{N} \sum_{n<N} T^n f_1 \dots T^{kn} f_k \Big\|_2 \leq C \min_i |||f_i|||_k .$$

Theorem. *Let $f_1, \dots, f_k \in L^\infty(\mu)$ satisfy $\|f_i\|_\infty \leq 1$ and let $p_1, \dots, p_k$ be integer polynomials.*
There exist ℓ and C such that

$$\limsup_{N\to+\infty} \Big\| \frac{1}{N} \sum_{n<N} T^{p_1(n)} f_1 \dots T^{p_k(n)} f_k \Big\|_2 \leq C \min_i |||f_i|||_\ell .$$

C and ℓ depend only on the polynomials.

The proofs use:

Hilbert space van der Corput Lemma Let $\{\xi_n\}$ be a sequence in a Hilbert space with $\|\xi_n\| \leq 1$ for every n. Then

$$\limsup_N \Big\| \frac{1}{N} \sum_{n<N} \xi_n \Big\|^2 \leq \limsup_H \frac{1}{H} \sum_{h<H} \Big(\limsup_N \Big| \frac{1}{N} \sum_{n<N} \langle \xi_{n+h} \mid \xi_n \rangle \Big| \Big) .$$

The linear case. The proof goes by induction.

$$\Big\| \frac{1}{N} \sum_{n<N} T^n f_1 \Big\|_2 \to \Big| \int f_1 \, d\mu \Big| = |||f_1|||_1$$

At each step we use van der Corput, Cauchy-Schwartz, Hölder and the inductive definition of the seminorms.
The polynomial case uses a form of the PET induction of Bergelson.

3. The structure theorem

The seminorms can be interpreted in terms of nilsystems.
We recall that (X, μ, T) is an ergodic system.

Theorem (H & Kra). *Let $k \geq 2$ be an integer, $f \in L^\infty(\mu)$ and $\epsilon > 0$.*
There exist a $(k-1)$-step nilsystem (Y, ν, S), a factor map $\pi \colon X \to Y$ and a smooth function h on Y with

$$\|h\|_\infty \leq \|f\|_\infty$$

and

$$\|\!|f - h \circ \pi|\!\|_k \leq \epsilon \ .$$

We do not give a proof of this theorem in these talks, but we give some indications of how the first step works.

Using the structure theorem.
For $f_1, \ldots, f_k \in L^\infty(\mu)$ we prove the convergence in $L^2(\mu)$ of the averages

$$\frac{1}{N} \sum_{n<N} T^n f_1 \ldots T^{kn} f_k \ .$$

We can assume that $\|f_i\|_\infty \leq 1$ for all i.
Let (Y, ν, S) be a $(k-1)$-step nilsystem, $p \colon X \to Y$ a factor map and let h_i, $1 \leq i \leq k$, be continuous functions on Y with

$$\|h_i\|_\infty \leq 1 \text{ and } \|\!|f_i - h_i \circ p|\!\|_k < \epsilon \text{ for every } i \ .$$

$$\limsup_N \Big\| \frac{1}{N} \sum_{n<N} T^n f_1 \ldots T^{kn} f_k - \big(\frac{1}{N} \sum_{n<N} S^n h_1 \ldots S^{kn} h_k \big) \circ p \Big\|_2$$
$$\leq C \sum_{i=1}^{k} \|\!|f_i - h_i \circ p|\!\|_k \leq Ck\epsilon \ .$$

As (Y, ν, S) is a nilsystem,

$$\frac{1}{N} \sum_{n<N} S^n h_1 \ldots S^{kn} h_k \text{ converges in } L^2(\nu)$$

thus

$$\Big(\frac{1}{N}\sum_{n<N} S^n h_1 \dots S^{kn} h_k\Big) \circ p \text{ converges in } L^2(\mu).$$

$\Big(\dfrac{1}{N}\displaystyle\sum_{n<N} T^n f_1 \dots T^{kn} f_k \colon N \geq 1\Big)$ is a Cauchy sequence. □

Nilsystems in ergodic Theory III

The Kronecker factor and the second seminorm

Henceforth, (X, μ, T) is an ergodic measure preserving system.

We recall the definition of the second seminorm:
for $f \in L^\infty(\mu)$,

$$|||f|||_2^4 = \lim_{N\to+\infty} \frac{1}{N}\sum_{n<N}\Big(\int f \cdot T^n f \, d\mu\Big)^2$$

This expression could be analyzed by using the correlation measure of f but we prefer to use another method.

1. The case of a rotation

Here (Z, m, T) is an ergodic rotation and $Tx = \alpha x$.
We recall that for $f \in L^\infty(m)$,

$$|||f|||_2^4 = \lim_{N\to+\infty} \frac{1}{N}\sum_{n=0}^{N-1}\Big(\int f \cdot T^n f \, dm\Big)^2$$

$$|||f|||_2^4 = \int_H f \otimes f \otimes f \otimes f \, dm_2$$

where

$$f \otimes f \otimes f \otimes f(x_0, x_1, x_2, x_3) = f(x_0) f(x_1) f(x_2) f(x_3) \ ,$$

H is the closed subgroup

$$H := \{(x_0, x_, x_2, x_3) \in Z^4 \colon x_0 x_1^{-1} x_2^{-1} x_3 = 1\}$$

of Z^4 and m_2 is its Haar measure.

Moreover,

$$|||f|||_2^4 = \int f(x) f(sx) f(tx) f(stx) \, dm(x) \, dm(s) = \sum_{\chi\in\widehat{Z}} |\widehat{f}(\chi)|^4$$

and $|||f|||_2 = 0$ iff $f = 0$:
If the system is a rotation, then $|||\cdot|||_2$ is a norm.

2. The second seminorm in the general case.

Definition.
If $\pi\colon (X,\mu,T)\to(Y,\nu,T)$ is a factor map and $f\in L^1(\mu)$,
$\mathbb{E}(f\mid Y)$ is the function on Y defined by

$$\int \mathbb{E}(f\mid Y)\cdot g\,d\nu = \int f\cdot g\circ\pi\,d\nu \text{ for all } g\in L^\infty(\nu)\ .$$

If $f\in L^p(\mu)$ then $\mathbb{E}(f\mid Y)\in L^p(\nu)$.

Definition. If $\mathcal{J}$ is an invariant σ-algebra on X, the conditionally independent product measure $\mu\times_{\mathcal{J}}\mu$ on X^2 is characterized by:

For $f,f'\in L^\infty(\mu)$,

$$\int f(x)f'(x')\,d\mu\times_{\mathcal{J}}\mu(x,x') = \int \mathbb{E}(f\mid\mathcal{J})\cdot\mathbb{E}(f'\mid\mathcal{J})\,d\mu\ .$$

Computing the second seminorm.
Let $\mathcal{I}(T\times T)$ be the $T\times T$-invariant σ-algebra of $(X\times X,\mu\times\mu,T\times T)$.

$$\begin{aligned}
\|\!|f|\!\|_2^4 &= \lim_{N\to+\infty}\frac{1}{N}\sum_{n<N}\Big(\int f\cdot T^nf\,d\mu\Big)^2\\
&= \lim_{N\to+\infty}\int f\otimes f\cdot\frac{1}{N}\sum_{n<N}(T^nf\otimes T^nf)\,d\mu\times\mu\\
&= \int \big(\mathbb{E}_{\mu\times\mu}\big(f\otimes f\mid\mathcal{I}(T\times T)\big)\big)^2\,d\mu\times\mu\\
&= \int f\otimes f\otimes f\otimes f\,d\mu_2
\end{aligned}$$

where μ_2 is the measure

$$\mu_2=(\mu\times\mu)\times_{\mathcal{I}(T\times T)}(\mu\times\mu)$$

on X^4.
This measure is related to the Kronecker factor.

The Kronecker factor.

Definition. The Kronecker factor (Z,m,S) of (X,μ,T) is the factor spanned by the eigenfunctions.
Z can be given the structure of a compact abelian group,
m is its Haar measure and T is the rotation given by some $\alpha\in Z$.

Let $\pi\colon X \to Z$ be the factor map.
The invariant functions of $(X \times X, \mu \times \mu, T \times T)$ are the functions of the form

$$F(x_0, x_1) = h\big(p(x_0)p(x_1)^{-1}\big)$$

where h is a function on Z.

Therefore, for $f \in L^\infty(\mu)$,

$$\mathbb{E}_\mu(f \mid Z) = 0 \Longrightarrow \mathbb{E}_{\mu\times\mu}\big(f \otimes g \mid \mathcal{I}(T \times T)\big) = 0 \text{ for every } g \in L^\infty(\mu)\ .$$

We get:

Proposition. *For $f \in L^\infty(\mu)$,*

$$|\!|\!| f - \mathbb{E}(f \mid Z) \circ \pi |\!|\!|_2 = 0\ .$$

In particular, $|\!|\!| f |\!|\!|_2 = 0$ if and only if $\mathbb{E}(f \mid Z) = 0$.

Let $f \in L^\infty(\mu)$ satisfies $\|f\|_\infty \le 1$.
There exist a rotation (Y, ν, S) where Y is a compact abelian Lie group, a factor map $q\colon Z \to Y$ and a smooth function h on Y with

$$\|h\|_\infty \le \|\mathbb{E}(f \mid Z)\|_\infty \quad \text{and} \quad \|\mathbb{E}(f \mid Z) - h \circ q\|_1 < \epsilon/8\ .$$

Since $\|\mathbb{E}(f \mid Z) - h \circ q\|_\infty \le 2$, we have $|\!|\!|\mathbb{E}(f \mid Z) - h \circ q|\!|\!|_2 < \epsilon$.
Let $p = \pi \circ q$. $p\colon X \to Y$ is a factor map and

$$\begin{aligned}\|f - h \circ p\|_2 &\le |\!|\!| f - \mathbb{E}(f \mid Z) \circ \pi |\!|\!|_2 + |\!|\!|\mathbb{E}(f \mid Z) - h \circ q|\!|\!|_2 \\ &= |\!|\!|\mathbb{E}(f \mid Z) - h \circ q|\!|\!|_2 < \epsilon\ .\end{aligned}$$

This proves the structure theorem for the second seminorm.

End of the second talk