Nilsystems in ergodic Theory IV

Conze-Lesigne equation

In the proof of the structure theorem, nilsystems are built by solving some functional equations.
Conze-Lesigne Equation is the simplest of these equations.
It is used in the proof of the structure theorem for the third semi-norm.We do not explain where this equation comes from, butonly how it is used in the construction of a 2-step nilsystem.

Let (Z, m, T) be an ergodic rotation:
Z is a compact abelian group, m its Haar measure and $Tx = \alpha x$.Let ρ be a (measurable) function on Z with values in the circle group $\mathcal{S}^1$.
We say that ρ satisfies the Conze-Lesigne equation if:For every $s \in Z$ there exist a function $f \colon Z \to \mathcal{S}^1$ and a constant c with

$$\text{(CL)} \qquad \frac{\rho(sz)}{\rho(z)} = c\,\frac{f(\alpha z)}{f(z)}\ .$$

(f and c depend on s.)Remark: we could only assume that h and c exist only for s in asubset of Z of positive measure.

Theorem (Conze & Lesigne)**.** *Assume that (Z, m, T) is an ergodic rotation on the Lie group Z and that $\rho\colon Z \to \mathcal{S}^1$ satisfies the Conze-Lesigne equation.Let $Z \times \mathcal{S}^1$ be endowed with the measure $\mu = m \times$ Lebesgue and with the transformation S given by*

$$S(z, u) = \big(Tz, \rho(z)u\big)\ .$$

Then:The system $(Z \times \mathcal{S}^1, \mu, S)$ is isomorphic in the measure theoretical sense to a 2-step nilsystem.

If we do not assume that the compact abelian group Z is a Lie group then the system is isomorphic to an inverse limit of a sequence of 2-step nilsystems.

Proof. We recall that $Tz = \alpha z$.
For every $s \in Z$ and every function $f \colon \mathbb{Z} \to \mathcal{S}^1$ satisfying the equation

$$\text{(CL)} \qquad \frac{\rho(sz)}{\rho(z)} = c\,\frac{f(\alpha z)}{f(z)}\ .$$

for some c, we define a measure preserving transformation $R_{s,f}$ of $Z \times \mathcal{S}^1$ by

$$R_{s,f}(z, u) = (sz, f(z)u)\ .$$

We remark that $S = R_{\alpha,\rho}$.The transformations $R_{s,f}$ where s and f satisfy (CL) for some c form a group G for the composition:

$$R_{s,f} \circ R_{t,g} = R_{st,h} \text{ where } h(z) = f(tz)g(z)$$

and (st, h) satisfy (CL) for some constant.

Algebraic properties of G.
The commutator of $R_{s,f}$ and $R_{t,g}$ is:

$$R_{1,q} \text{ where } q(z) = \frac{f(tz)g(z)}{f(z)g(sz)} \ .$$

As (s, f) and (t, g) satisfy (CL):

$$\frac{\rho(sz)}{\rho(z)} = c\,\frac{f(\alpha z)}{f(z)} \text{ and } \frac{\rho(tz)}{\rho(z)} = c'\,\frac{g(\alpha z)}{g(z)}$$

we have $q(\alpha z) = q(z)$.
By ergodicity of (Z, T), the the function q is constant.Then the commutator subgroup G_2 of G is included in
$\{R_{1,q} \colon q \text{ constant}\}$ and thus it is included in the center of G:G is 2-step nilpotent.

The map

$$p\colon R_{s,f} \mapsto s$$

is a group homomorphism from G to Z, and p is onto by hypothesis.
The kernel of this homomorphism is the family of transformations $R_{1,f}$ where f satisfies

$$1 = c\,\frac{f(\alpha z)}{f(z)} \text{ for some } c$$

that is, where f is an affine function:

$$f(z) = \kappa\chi(z) \text{ where } |\kappa| = 1 \text{ and } \chi \in \widehat{Z}$$

and thus the kernel of p is isomorphic to $\mathcal{S}^1 \times \widehat{Z}$.

The topology of G.
Let the group G of transformations of $Z \times \mathcal{S}^1$ be endowed with the topology of the convergence in probability:
R_{s_i,f_i} converges of $R_{s,f}$ in G iff $s_i \to s$ in Z and $f_i \to f$ in $L^1(Z)$.G is a Polish group.The group homomorphism p from G onto Z is continuous.The isomorphism $\ker(p) \cong \mathcal{S}^1 \times \widehat{Z}$ is a homeomorphism.As Z and $\mathcal{S}^1\widehat{Z}$ are Lie groups,
G is a 2- step nilpotent Lie group.

The subgroup $\Gamma = \{R_{1,\chi} \colon \chi \in \widehat{Z}\} \cong \widehat{Z}$ is discrete and cocompact in G.
Γ is the stabilizer of the point $(1,1)$ of $Z \times \mathcal{S}^1$.We can identify the nilmanifold $X = G/\Gamma$ with $Z \times \mathcal{S}^1$.
The Haar measure on X corresponds to the measure $\mu = m \times$ Lebesgue on $Z \times \mathcal{S}^1$.
The transformation S of $Z \times \mathcal{S}^1$ corresponds to the translation by the element $R_{1,\rho}$ of G. $(Z\times\mathcal{S}^1, \mu, S)$ is isomorphic to a 2 step nilsystem. $\square$

Nilsystems in ergodic Theory V

Cubes in nilmanifolds

Cubes of dimension 2

Let G be group and let $G^{[2]}$ be the subgroup of G^4:

$$G^{[2]} = \mathrm{gp}\big\{(x,x,x,x),\ (1,1,s,s),\ (1,t,1,t) \colon a,b,c \in G\big\}\ .$$

We have

$$G^{[2]} = \big\{(x, xs, xt, xstu) \colon x,s,t \in G,\ u \in G_2\big\}\ .$$

Each coordinate of an element g of $G^{[2]}$ is completely determined by the three others iff G is abelian.Let (Z,m,T) be a rotation on the compact abelian group Z and f a function on G. We recall that

$$\|\!|f|\!\|_2^4 = \int f\otimes f\otimes f\otimes f\, dm^{[2]}$$

where $m^{[2]}$ is the Haar measure of $Z^{[2]}$.

The general case.

Let $k \geq 2$ be an integer and G a group.
The points of G^{2^k} are written as

$$g = (g_\epsilon \colon \epsilon \in \{0,1\}^k)$$

For $g \in G$ and $F \subset \{0,1\}^k$, g^F is the element of G^{2^k} given by

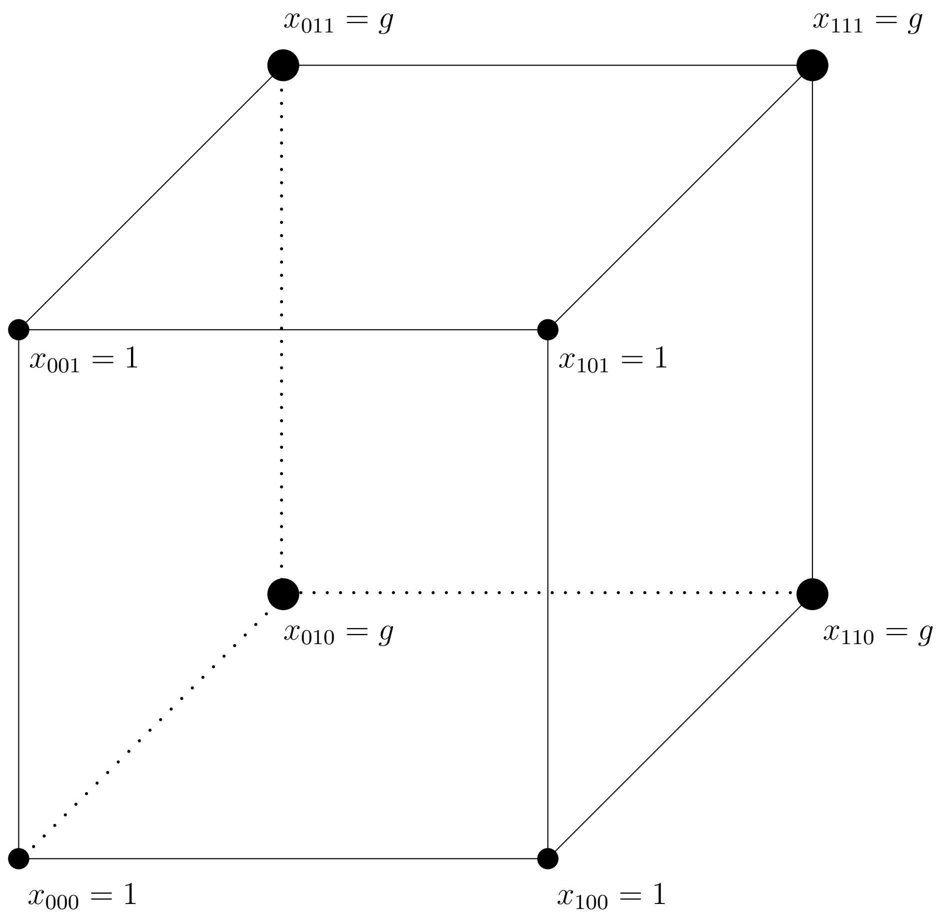
$$\text{for } \epsilon \in \{0,1\}^k, \quad (g^F)_\epsilon = \begin{cases} g & \text{if } \epsilon \in F\ ; \\ 1 & \text{otherwise.} \end{cases}$$

It is also convenient to identify $\{0,1\}^k$ to the set of vertices of the unit cube in $\mathbb{R}^k$.
For $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$, the subset

$$F = \big\{\epsilon \in \{0,1\}^k \colon \epsilon_{i_1} = \epsilon_{i_2} = \cdots = \epsilon_{i_\ell} = 1\big\}$$

is called a upper face of codimension 1.

The face $F = \left\{\epsilon \in \{0,1\}^3 \colon \epsilon_2 = 1\right\}$ **of codimension** 1
and $x = g^F = (1,1,g,g,1,1,g,g) \in G^{2^3}$ **.**



Definition. The group of cubes of dimension k is the subgroup $G^{[k]}$ of G^{2^k} spanned by:

$$\left\{g^{\Delta} := (g, g, \dots, g) \colon g \in G\right\}$$
$$\text{and } \left\{g^F \colon g \in G \text{ and } F \text{ is a face of codimension 1 of } \{0,1\}^k\right\} .$$

These cubes should be called parallelepipeds: their edges are not equal… **Another description.** Let $F_1, \dots, F_{2^k}$ be an enumeration of all upper faces of $\{0,1\}^k$. Then each element of $G^{[k]}$ can be written in an unique way as

$$g_1^{F_1} g_2^{F_2} \dots g_{2^k}^{F_{2^k}}$$

where $g_i \in G_j$ if F_i is of codimension j.

Proposition. *Let $X = G/\Gamma$ be a nilmanifold.*

The $G^{[k]}$ is a closed subgroup of G^{2^k}, hence a Lie group and

$\Gamma^{[k]} := \Gamma^{2^k} \cap G^{[k]}$ is a discrete and cocompact subgroup of $G^{[k]}$.

The nilmanifold $X^{[k]} := G^{[k]}/\Gamma^{[k]}$ is called the nilmanifold of cubes of dimension k of X.

Idea of the proof. By induction. Let H be the normal subgroup of G^{2^k}

$$H = \left\{g \in G^{[k]} \colon g_\epsilon g_\eta^{-1} \in G_2 \text{ for all } \epsilon, \eta \in \{0,1\}^k\right\} .$$

Identifying $G^{2^{k+1}} = G^{2^k} \times G^{2^k}$, we have

$$G^{[k+1]} = \left\{(h, h') \in G^{[k]} \times G^{[k]} \colon h = h' \bmod H\right\}$$

and a similar formula holds for $\Gamma^{[k+1]}$.

Let $\mu^{[k]}$ be the Haar measure of $X^{[k]}$.
Each projection of $\mu^{[k]}$ on X is equal to the Haar measure μ of X.For $f \in L^\infty(\mu)$ we define:

$$|||f|||_k = \left(\int f \otimes f \otimes \dots \otimes f \, d\mu^{[k]}\right)^{1/2^k}$$

(the integral is ≥ 0).
We have an inequality similar to the Cauchy-Schwartz-Gowers inequality: If f_ϵ, $\epsilon \in \{0,1\}^k$, belong to $L^\infty(\mu)$ we have

$$\left|\int \bigotimes_{\epsilon \in \{0,1\}^k} f_\epsilon \, d\mu^{[k]}\right| \leq \prod_{\epsilon \in \{0,1\}^k} |||f_\epsilon|||_k \ .$$

$\|\!|\cdot|\!\|_k$ is a seminorm on $L^\infty(\mu)$.
(Also on $L^{2^k}(\mu)$)

Proposition. *Let $\tau \in G$ be such that the translation $T\colon x \mapsto \tau \cdot x$ on X is ergodic.*
Then the seminorm defined on $L^\infty(\mu)$ by

$$|\!|\!| f |\!|\!|_k = \Big(\int f \otimes f \otimes \cdots \otimes f \, d\mu^{[k]} \Big)^{1/2^k}$$

coincides with the seminorm associated to the system (X, μ, T) as explained in the preceding lecture.

In the case of nilsystems, these seminorms are geometric and not dynamical objects.

Theorem. *The following properties are equivalent:*

1) G is $(k-1)$-step nilpotent.

2) Each coordinate of $X^{[k]}$ is determined by the 2^k-1 other coordinates.

3) $|\!|\!| \cdot |\!|\!|_k$ is a norm on $L^\infty(\mu)$ (and on$L^{2^k}(\mu)$).

Henceforth $X = G/\Gamma$ is a $(k-1)$-step nilmanifold.The completion of $L^\infty(\mu)$ for the norm $|\!|\!| \cdot |\!|\!|_k$ is not easy to describe; it consists in distributions.

Dual functions on a nilmanifold

We know that the first coordinate of $X^{[k]}$ is a function of the others:
Let X^* be the projection of $X^{[k]}$ on X^{2^k-1}.
There exists a smooth map $\Phi\colon X^* \to X$ with

$$X^{[k]} = \big\{ (x^*, \Phi(x^*)) \colon x^* \in X^* \big\} \ .$$

For every $x \in X$, the fiber $\Phi^{-1}\{x\}$ is a nilmanifold.
Let m_x be its Haar measure. For $f \in L^{2^k-1}(\mu)$, the dual function ${}_kf$ of f is the function on X given by

$${}_kf(x) = \int_{\Phi^{-1}\{x\}} \underbrace{f \otimes f \otimes \cdots \otimes f}_{2^k-1 \text{ times}} \, dm_x \ .$$

For $f \in L^{2^k-1}(\mu)$, ${}_k f$ is continuous on X. If (X,μ,T) is an ergodic $(k-1)$ step nilsystem, can be defined in dynamical terms:

$$ {}_k f(x) = \lim_{N\to+\infty} \frac{1}{N^k} \sum_{n_1,\dots,n_k<N} \prod_{\substack{\epsilon\in\{0,1\}^k\\ \epsilon\neq\mathbf{0}}} f\big(T^{n_1\epsilon_1+\cdots+n_k\epsilon_k}x\big) $$

The dual norm

Theorem. *Let B be the unit ball of the dual space of $(L^\infty(\mu), \|\!|\cdot|\!\|_k)$.*
Then B is the closed convex hull in $L^{2^k/(2^k-1)}(\mu)$ of

$$ \big\{ {}_k f \colon f\in L^\infty(\mu),\ \|\!|f|\!\|_k \le 1\big\}\ . $$

Theorem. *Every sufficiently regular function f on X belongs to the dual space of $(L^\infty(\mu), \|\!|\cdot|\!\|_k)$.*
This means that there exists a constant $C>0$ such that

$$ \Big|\int f\,h\,d\mu\Big| \le C\cdot \|\!|h|\!\|_k \text{ for every } h\in L^\infty(\mu)\ . $$

Let $\|\!|f|\!\|_k^*$ be the dual norm.

Application to nilsequences

Let (X,T) be a $(k-1)$-step nilsystem, f a continuous function on X and $x_0\in X$.
Then the sequence

$$ \big(f(T^n x_0)\colon n\in\mathbb{Z}\big) $$

is called a (basic) $(k-1)$-step nilsequence.
Is is called a smooth $(k-1)$-step nilsequence is f is smooth.We can always reduce to the case that (X,T) is ergodic.

The same sequence can be defined as a nilsequence in different ways. However we have:Let (X,T) and (X',T') be ergodic $(k-1)$-step nilsystems, f a continuous function on X, f' a continuous function on X', $x_0\in X$ and $x_0'\in X'$.If $(f(T^n x_0) = f'(T'^n x_0')$ for every n then $\|\!|f|\!\|_k = \|\!|f'|\!\|_k$.If moreover f and f' are smooth then $\|\!|f|\!\|_k^* = \|\!|f'|\!\|_k^*$.The norm and the dual norm are attached to the sequences themselves, independently of their construction.

End