Arithmetic Progressions in Sets of Fractal Dimension

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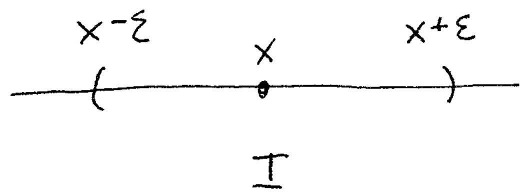
Continuous analogues of Szemerédi's theorem?

$E \subseteq [0,1]$, $|E| > 0$, must E contain k-term AP $\{x, x+r, \ldots, x+(k-1)r\}$ with $r \neq 0$?

-Easy:
Use Lebesgue Density Thm



$$|E \cap I| \geq 0.99I$$

Hard question (Erdős): $E \subset [0,1]$, $|E| > 0$, must E contain an affine copy of $\{a_n\}_{n=1}^{\infty}$ (infinite sequence)? E.g. $\{2^{-n}\}_{n=1}^{\infty}$?

WLOG $a_n \to 0$.
If $\frac{a_n}{a_{n-1}} \to 1$ as $n \to \infty$ (e.g. $a_n = \frac{1}{n}$) then false (Falconer).
More examples: Kolountzakis, Bourgain

Easier question: $E \subset [0,1]$, $\dim E = \alpha$ (α close to 1)
Must E contain $\{x, x+r, \ldots, x+(k-1)r\}$?
No (Keleti 1998)

Theorem (Łaba-Pramanik) $E \subset [0,1]$, E supports a probability measure such that
(A.) $\mu((x, x+\epsilon)) \leq c_1 \epsilon^{\alpha}$, $\epsilon > 0$

(B.) define $\widehat{\mu}(k) = \int_0^1 e^{-2\pi i k x}\, d\mu(x)$. Then $|\widehat{\mu}(k)| \leq c_2 |k|^{-\beta/2}$, $k \neq 0$, some $\beta > \frac{2}{3}$.

If α is close enough to 1 (depending on c_1, c_2, β) then E contains a 3-AP.

Discussion:

(A.) $\dim(E) = \sup\{\alpha \colon \exists\, \mu \text{ supported on } E \text{ such that (A.) holds}\}$ (Frostman)

(B.) Fourier Dimension:

$$\dim_F E = \sup\{\beta \in [0,1) \colon \exists\ \mu \text{ supported on } E \text{ such that (B.) holds}\}$$

$$\dim_F E \leq \dim E$$

Ineq. can be sharp, eg. $\frac{2}{3}$ Cantor set has dim $= \frac{\log 2}{\log 3}$, $\dim_F = 0$

Salem sets: $\dim_F E = \dim E$. Other examples: Kahane, Kaufman

Our Conditions? Salem-OK (small modifications)
Kahane–(A fails) use different argument.

New Construction (Ł-P): given $0 < \beta < \alpha < 1$, $c_1 > 1$, $c_2 > 0$ we can find a set E, measure μ, such that (A), (B) holds.

Outline of proof: $\Lambda(\mu, \mu, \mu) = \sum_{k \in \mathbb{Z}} \widehat{\mu}(k)^2 \widehat{\mu}(-2k)$

(1) If $\Lambda > 0$ then E contains 3-APs. If μ abs. cts., density f

$$\Lambda = \frac{1}{2} \int \int f(x) f(y) f(\frac{x+y}{2})\, dx\, dy$$

In general: define ν on $[0,1]^2$ such that ν supp on $X = \{(x,y) \colon x, y, \frac{x+y}{2} \in E\}$

$$\nu([0,1]^2) = \Lambda(\mu, \mu, \mu) > 0$$
$$\nu(\{(x,x) \colon x \in [0,1]\} = 0$$

(2) If μ abs. cts., density f, $0 \leq f \leq \mathcal{M}$

$$\lambda(\mu, \mu, \mu) > c(M)$$

Proof: follow proof of Roth's theorem

(3) Restriction estimate for fractal sets
Mockenhaupt 2000, Ł-P

If μ obeys (A)-(B) then

$$\left(\int |f|^2\, d\mu\right)^{1/2} \leq c \left\| \widehat{f} \right\|_{l^p(\mathbb{Z})},$$
$$p = \frac{2(\beta + 4(1-\alpha))}{\beta + 8(1-\alpha)}$$

c depends ony on c_1, c_2

Main Argument: (follows Green, Green-Tao, Tao-Vu)

$$\mu = \mu_1 + \mu_2$$

μ_1 abs. cts., density $f \leq M$
$\widehat{\mu_2}(k)$ small
Construct μ_1: $\mu_1 = \mu * K_{2N}$

$$K_{2N} = \frac{1}{2N+1} \left| \sum_{k=-N}^{N} e^{2\pi i k x} \right|^2$$

μ_1 has bdd density–follows from restriction

$$\Lambda(\mu, \mu, \mu) = \Lambda(\mu_1, \mu_1, \mu_1) + \Lambda(\mu_2, \mu_1, \mu_1) + \dots$$
$$\Lambda(\mu_1, \mu_1, \mu_1) > c(M)$$

other terms small because $\widehat{\mu_2}$ small