Correspondence principle and finitary ergodic theory, I

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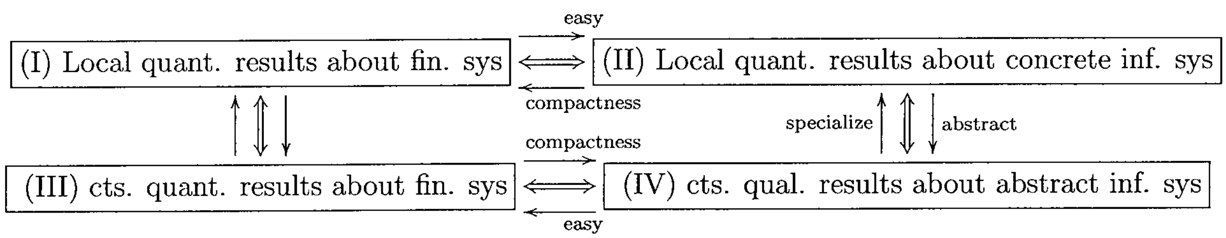
Correspondence Principle:



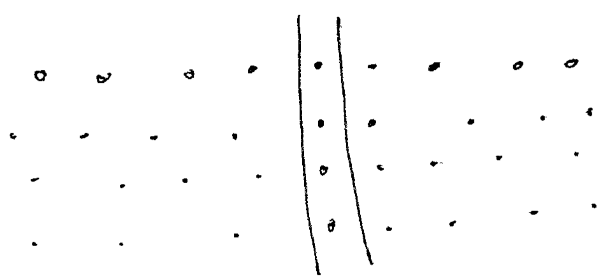
(1) Corresponence Principle in Ramsey Theory

van der Waerden's Thm: (II) If the integers $\mathbb{Z}$ are coloured by c colors, $k \geq 1$, then $\exists$ k-term AP which is monochromatic.

(I) $\forall$ $c, k \geq 1$, $\exists$ $N_0(c,k)$ s.t. if $\{-N, \ldots, N\}$ are coloured by c colours, then $\exists$ k-term AP which is monochromatic.

(II) $\Rightarrow$ (I): compactness + contradiction

Assume (II), suppose (I) false. $\exists$ $c, k \geq 1$ s.t. $\forall$ N $\exists$ colouring of $\{-N, \ldots, N\}$ into c colours with no monochromatic k-term AP. In particular, $\exists$ $N_n \to \infty$ and colourings $c_n \colon \{-N_n, \ldots, N_n\} \to \{1, \ldots, c\}$ with no k-term APs. Take the limit weak topology



Weak sequential compactness: given any sequence of finite colourings with c colours on $\{-N, \ldots, N\}$ $\exists$ subsequence of colourings which converge weakly to an infinite colouring.

Proof. Arzela-Ascoli argument (diagonal argument) $\square$

van der Waerden's Thm (III): If X compact top. space, $X \subseteq V_1 \cup \ldots \cup V_c$ is an open cover by c open sets.

$T \circlearrowright X$ homeomorphism.

$\exists$ $x \in X$, $r > 0$, $x, T^r x, T^{2r} x, \ldots, T^{(n-1)r} \in V_i$ for some i.

$x_0 \colon \mathbb{Z} \to \{1, \ldots, c\}$

$x_0 \in c^{\mathbb{Z}} = \{1, \ldots, c\}^{\mathbb{Z}}$

$c^{\mathbb{Z}} \supseteq V_j \colon = \{x \in c^{\mathbb{Z}} \colon x(0) = j\}$
$T \colon c^{\mathbb{Z}} \to c^{\mathbb{Z}}$ left shift
$Tx(n) = x_{n+1}$
x_0 contains a monochromatic AP
$\Rightarrow \exists\ n \in \mathbb{N}$ s.t. $T^n x_0, T^{n+r} x_0, \ldots, T^{n+(k-1)r} x_0 \in V_i$
$X = \overline{\{T^n x_0 \colon n \in \mathbb{Z}\}}$

(2) Correspondence Principle in theory of finitely generated groups

G gen. by finite set S

$B_{r,s,G} = \{$words of finite length r using $S \cup S^{-1}$ as an alphabet$\}$

(G, S) has poly. growth if $\exists\ k,\ d \geq p$ s.t. $\forall\ r,\ |B_{r,S,G}| \leq Kr^d$.

Easy: $\forall$ virtually nilpotent f.g. groups have poly. growth

Gromov Thm (II): every f.g. group of polynomial growth is virtually nilpotent

$\Updownarrow$ $\uparrow$ easy $\downarrow$ compactness + contradiction

Gromov's Thm (I): $\forall\ s, K, d \geq 1$, $\exists\ N_0(s,K,d)$, $R_0(s,K,d)$ s.t.
(G, S) $|S| \leq s$ $\forall\ r \leq R_0$, $|B_{r,S,G}| \leq Kr^d$
$\Rightarrow G$ virt. nilp.
Furthermore, G contains a subgroup G' of step $\leq N_0$ and index $\leq N_0$

$(G_n, S_n) \to \infty$ $k_n \to \infty \overset{\text{limit}}{\longrightarrow} (G, S)$
poly growth up to scale R_n but $\forall\ N$, G_n does not have a nilp. subgroup of index $\leq N$ if n is suff larg dep. on N

Since G_n has s generators, one can write $G = \mathbb{F}_s/\Gamma$. Then we use weak compactness to extract a subsequence for which Γ_n converges to Γ. If we set $G = \mathbb{F}_s/\Gamma$, then G will be of polynomial growth, and thus have a virtually nilpotent subgroup by Gromov's theorem. The property of being virtually nilpotent can be finitely presented, and so also holds for some G_n, a contradiction.