(1) Differential Equations and Operators, Linear and Nonlinear.

Might consider Laplace equation $\Delta u = 0$, and Poisson eq. $\Delta u = f$, where

$$\Delta = \operatorname{div}\operatorname{grad} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

and more generally $Pu = 0$, $Pu = f$.

May also consider differential operator Δ, or P. Then solvability of the equation is equivalent to asking: Is f in the image (or range) of the operator?

Likewise, questions ~~abt~~ about the number & nature of sol's u or f, (e.g. regularity properties — is sol. in C^2? H^2? etc) can be reformulated as mapping properties of the operator or solution operator.

An operator being linear or nonlinear refers to being linear or not in u and its derivatives, not to the nature of the coeff's. For ex., presence of a term $e^x \frac{\partial u}{\partial x_i} \not\Rightarrow$ nonlinear, but e^u or $e^u \frac{\partial u}{\partial x_i}$ does make the eq. & the corresponding operator nonlinear.

$\boxed{EX}$ ~~u_t~~ $u_t + uu_x = 0$ Burger's Eq

$$\frac{\partial u}{\partial t} - \Delta(u^\gamma) = 0, \quad \gamma > 1 \qquad \text{Porous Medium}$$

These are nonlinear.

Predominant behavior is determined by the highest order terms, so nonlinear equations & operators are arranged in a hierarchy according to how bad the nonlinearity is in the highest order terms.

$\boxed{EX}$ Reaction Diffusion: $\frac{\partial u}{\partial t} - \Delta u = f(u)$. semilinear.

$\boxed{EX}$ Minimal surface.

$$\operatorname{div}\left(\frac{\nabla u}{(1+\|\nabla u\|^2)^{1/2}}\right) = 0.$$

Quasilinear.

If products of the highest order terms occur, then fully nonlinear.

Coeff's of the highest order terms are also very important. For ex., if on a compact set, suppose we have information about $f'(x)$ everywhere, & f somewhere. This actually f controlled everywhere (by Mean Value Theorem). But if coeff. of f' vanishes or unbdd, lose control of f' and f.

$\boxed{\text{Ex}}$ $x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, in a region intersecting $\{x=0\}$,

or $\dfrac{1}{1+x^2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, in a region where x unbdd.

Problematic, because as $x \to 0$ or $|x| \to \infty$, lose information about x-deriv's.
~~The~~ second is elliptic but not uniformly elliptic: $\Lambda(x)/\lambda(x)$ unbdd.
First is degenerate.

$\boxed{\text{Ex}}$ Minimal surface eq; phenomenon is similar to second example above, except $\dfrac{1}{(1+\|\nabla u\|^2)^{1/2}}$ instead of $\dfrac{1}{1+x^2}$. Even in bdd domain, this coeff could be unbdd.

$\boxed{\text{Ex}}$ Porous Medium: High order terms $\dfrac{\partial u}{\partial t} - \partial u^{\partial-1}\dfrac{\partial^2 u}{\partial x^2}$
Degenerate, because can have $u=0$.

In the second 2 ex's, the nonlinearity adds another wrinkle: Need the solution to bound the ~~deriv~~ coeff's, but need to bound coeff's to find solution. Circular!

Recall from Calculus the definition of $f'(x_0)$ (and df_{x_0} in higher dim's) — existence of a linear approximation with error $\to 0$ faster than linearly.

Similarly here, if $P: \mathcal{U} \to \mathcal{G}$, then $dP_u : \mathcal{F} \to \mathcal{G}$

$$\begin{array}{c} \mathcal{U} \\ \cap \\ \mathcal{F} \end{array}$$

And as in Calculus, if differentiable, this deriv. is linear approx. can be calculated easily.

$$dP_u(v) = \frac{d}{d\sigma}\bigg|_{\sigma=0} P(u+\sigma v).$$

$\boxed{\text{Ex}}$ Above, we had reaction diffusion equation; corresponding operator is $P(u) = \frac{\partial u}{\partial t} - \Delta u - f(u)$ (of interest would be $P(u) = 0$, then).

$$P(u+\sigma v) = \frac{\partial u}{\partial t} + \sigma \frac{\partial v}{\partial t} - \Delta u - \sigma \Delta v - f(u+\sigma v)$$

$$\frac{d}{d\sigma} P(u+\sigma v) = \frac{\partial v}{\partial t} - \Delta v - f'(u+\sigma v)\cdot v$$

$$\Rightarrow \frac{d}{d\sigma}\bigg|_{\sigma=0} P(u+\sigma v) = \frac{\partial v}{\partial t} - \Delta v - f'(u)\cdot v$$

linear equation.

$\boxed{\text{EX}}$ (Exercise) Find linearization at u for PM equation.

$$dP_u(v) = \frac{\partial v}{\partial t} - \gamma u^{\gamma-1}\Delta v + \{\text{lower order}\}.$$

Reminder: $\frac{\partial u}{\partial x_i}\frac{\partial^2 v}{\partial x_j \partial x_K}$ second order term,

$\frac{\partial v}{\partial x_i}\frac{\partial^2 u}{\partial x_j \cdot \partial x_K}$ a first order term.

Degenerate? Depends on u.

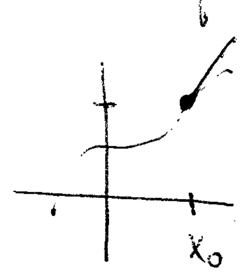
As in Calculus, to study f via its linearization, ie to study solvability or invertibility properties, supposed to linearize at a solution, or approximate solution (eg Newton's Method).

$\boxed{\text{EX}}$ $2x - 1 + \sin x = 10$ solvable?

$f(x) = 2x - 1 + \sin x$. Is 10 in the image?

$df_{x_0} = 2 + \cos x_0$: <u>always</u> invertible.

$\Rightarrow$ ~~If~~ the equation is solvable, then uniquely solvable in a small nbhd.



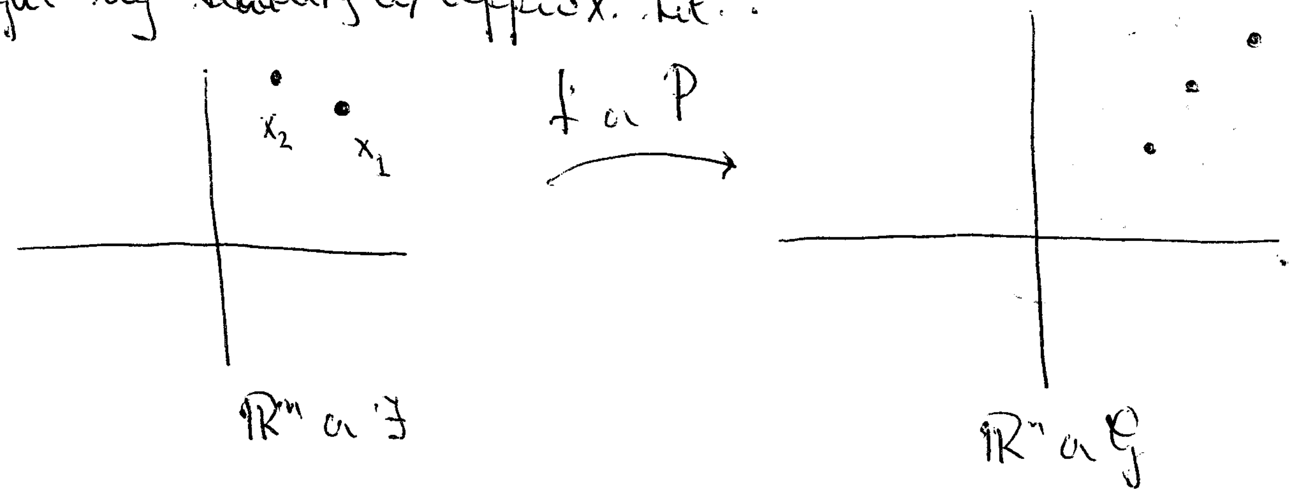
But it doesn't give existence.

$\boxed{\text{EX}}$ (Exercise) $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x^2+y^2, 3x-y)$

Find exact image. Do linear analysis.

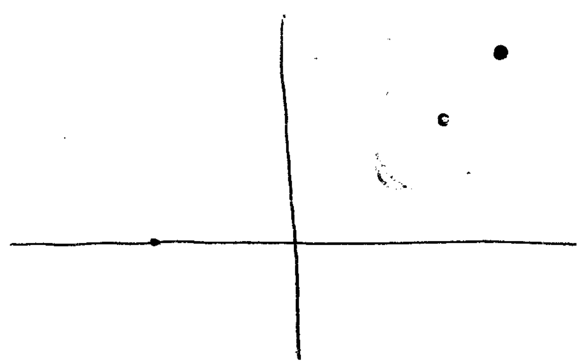
Inverse Fcn. Th. and Implicit Fcn Theorem are powerful theorems relating f to its linearization, ie, study mapping properties of f via those of df_{x_0}, but start w/ point in domain, ie, supposed to linearize around solution... but that's what we're trying to find.

Might try starting w/ approx. sol.:



Solve $Pu = C$.

More realistically



Radius shrinks.

Would need uniform estimates to get uniform control of radii.

Differential equation, so may be in $C^k(\Omega)$, for ex.:

- $\|u\|_{C^k} = \sup_{x\in\Omega} |u| + \dots + \sup_{x\in\Omega} \|\nabla^k u\|$.

(Or could be $H^k(\Omega)$ - meaning up to K derivs in L^2, ie, an integration-based space.)

In general, $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ and $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$.

Some Techniques for Nonlinear Eqs

- Iteration
- Fixed Pt. Theorems
- Inverse, Implicit Fcn Th.
- Continuity Method
- Compactness Theorems

Typical Issue: Need uniform estimates. For ex., unit ball → unit ball? Is a map a Contraction Mapping? All require uniform estimates

An example of recasting a differential equation (here an ODE) as a fixed point problem:

EX Solve $u'(t) - u(t) = f(u(t))$.

Can ~~we~~ solve $u'(t) - u(t) = f(v(t))$:

$$\mathcal{P}(v) = \int_0^t e^{t-s} f(v(s))\,ds + Ce^t, \text{ solution operator.}$$

A solution to the original equation occurs when $\mathcal{P}(u) = u$, ie, when $\mathcal{P}$ has a fixed point. (Maybe find by iteration?)

Here, we "froze" the inhomogeneous term and then used ~~the~~ knowledge of <u>linear</u> equations. Similarly in the quasilinear case — may freeze the coeff's at v, produce a linear equation which we can (hopefully) solve, + then iterate or apply another technique.

Example of finding these uniform bounds. Suppose in $C^k(\Omega)$, ie, spaces based on differentiation.

Maximum Principle:

If a max. occurs at $p \in \Omega$, then $\frac{\partial^2 u}{\partial x_i^2} \le C$,

so $\Delta u(p) \le C$, (or $Pu \le C$ at p for P = elliptic).

Suppose we can show

$$\Delta u \geq \dots \geq Au - B, \quad \text{with } A > 0, \ \forall x \in \Omega.$$

Then at max p, if in interior, $0 \geq \Delta u(p) \geq Au(p) - B$,

which implies $\quad u(p) \leq \cancel{A/B}\ B/A$

$$\text{Then} \quad u(x) \leq u(p) \leq \cancel{A/B}\ B/A \quad \forall x \in \Omega.$$

Likewise, to bound, for ex., first derivs: Apply same idea to $\partial u/\partial x_j$? More likely: $v = \sqrt{1 + \|\nabla u\|^2}$ or $v = 1 + \|\nabla u\|^2$.

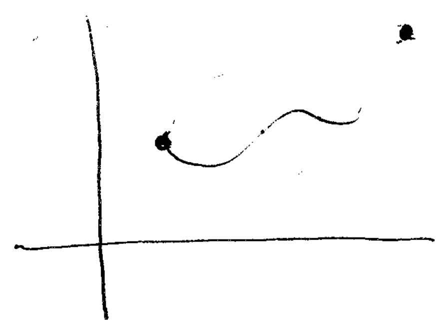
Maybe A = speed of wave propagation, scalar curvature, etc., depending on how the problem arose

Related to the idea of taking derivative, have:

(2) Variational Problems

Many PDE's that describe physical phenomena arise this way; a physical system will try to find the configuration of least energy ("energy" being defined according to the problem).

$\boxed{\text{EX}}$ Problem: Find the curve which minimizes length between 2 pts:



Makes sense to restrict to graphs. If also C^1,

$$l(f) = \int_a^b \sqrt{1+f_x^2}\, dx.$$

If f a minimizer, should have $l(f) \le l(f+tg)$ $\forall t, g$. So choose any g of compact support — thus $f+tg$ a curve between same endpts. If $l(f) \le l(f+tg)$, then should have $\left.\frac{d}{dt}\right|_{t=0} l(f+tg) = 0$. Calculate:

$$\cancel{\frac{d}{dx}}\, l(f+tg) = \int_a^b \sqrt{1+(f_x+tg_x)^2}\, dx$$

$$\frac{d}{dt}\, l(f+tg) = \int_a^b \frac{1}{2}\left(1+(f_x+tg_x)^2\right)^{-1/2} \cdot 2\,(f_x+tg_x)^1 g_x \, dx$$

$$\left.\frac{d}{dt}\right|_{t=0} l(f+tg) = \int_a^b \frac{f_x g_x}{(1+f_x^2)^{1/2}}\, dx.$$

Now use integration by parts to express as

$$\int_a^b (\dots)\, g\, dx. \quad \text{Then}$$

$$\left.\frac{d}{dt}\right|_{t=0} l(f+tg) = \int_a^b \frac{f_x}{(1+f_x^2)^{1/2}}\, g_x\, dx = -\int_a^b \frac{d}{dx}\left(\frac{f_x}{(1+f_x^2)^{1/2}}\right) g(x)\, dx.$$

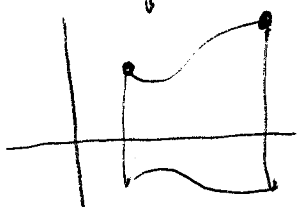
This should moreover be zero for all such choices of g.

That can only happen if the first factor is zero.

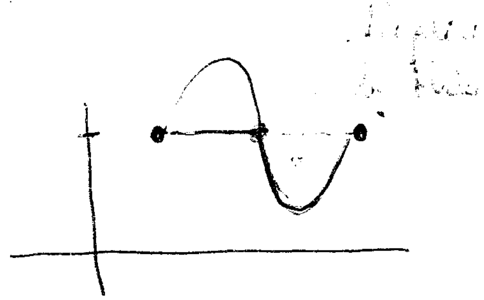
$$\Leftrightarrow \quad \frac{d}{dx}\left(\frac{f_x}{(1+f_x^2)^{1/2}}\right) = 0, \quad \text{or} \quad f_{xx} = 0.$$

Straight line. Euler-Lagrange equation

$\boxed{\text{EX}}$ (Exercise) For a curve between 2 fixed pts, ($f(x) > 0 \ \forall x \in [a,b]$) revolve around axis to generate a surface of revolution. Find the EL equation that describes the least area choice



Observation:



There would never be any reason to go above the horizontal.

Replace one segment - no longer C^1.

We can expand to allow such fcns ~~...~~ by using integration by parts, exactly as we did above. Namely, suppose

$u \in C^1(\Omega)$, $v \in C_0^1(\Omega)$. Then

$$\int_\Omega \frac{du}{dx_j} v\, dx = -\int_\Omega u \frac{dv}{dx_j}\, dx.$$

So if g satisfies $\displaystyle\int_\Omega g v\, dx = -\int_\Omega u \frac{dv}{dx_j}\, dx \quad \forall v \in C_0^1(\Omega)$,

Call g the weak deriv of u.

$H^k(\Omega) = \{ u$ and up to k weak deriv's in $L^2(\Omega)\}$. Sobolev space

Why introduce spaces based on integration? ie, to solve differential equations, it would seem best to work in spaces based on differentiation.

(1) Integration-based spaces bigger. May be easier to look for solution, ~~eg to $\Delta u = C$~~ in larger space (eg to $\Delta u = C$ in $L^2(\Omega)$) & prove regularity results after the fact.

(2) $L^2(\Omega)$ and the Sobolev spaces are Hilbert spaces. Can use all the inner product machinery such as orthogonal projections, maybe even O.N. basis.

(3) The origin of the problem may be such that integration, integration by parts, etc is natural, as was the case above.

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(3) Special solutions and their uses.

Consider $Pu = 0$ in $\mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$. $A: \mathbb{R}^n \to \mathbb{R}^n$ linear transformation. Say that A is a symmetry of the operator or equation if whenever $u(x)$ is a solution, so is $v(x) = u(Ax)$.

Might guess or determine, using A) direct calculation in lower dim., B) intuition based on physical or geometric origin of the problem, or C) having seen a similar problem, that P has a certain group of symmetries. Then may use this to produce an explicit solution.

$\boxed{\text{Ex}}$ The orthogonal group is a symmetry of Δ. (Check by direct calculation.) Use to ~~find sol~~ reduce to ODE: Choose $x_0 \in \mathbb{R}^n$. The orbit of x_0 under $O(n)$ is a sphere. The sphere is characterized by a single number, its radius.

$\Rightarrow$ Look for solutions $u(x) = u(x_1, \ldots, x_n) = v(r(x))$, where $r(x) = \|x\|$ ~~$\sqrt{\;}$~~. Then u satisfies the original PDE $\Leftrightarrow$ v solves an ODE. Here, and in favorable cases, this ODE can be solved explicitly. Then

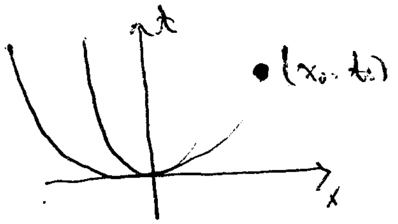
$$u(x) = \begin{cases} C \log \|x\|, & n = 2 \\ C \dfrac{1}{\|x\|^{n-2}}, & n > 2. \end{cases}$$

removable singularities

[illegible]

$\boxed{\text{EX}}$ Heat equation: $\frac{\partial u}{\partial t} - \Delta u = 0$

By a variety of methods, including guessing based on Laplace, deduce that P has parabolic symmetry.



Each orbit characterized completely by $\sigma = \frac{x^2}{t}$ or $\sigma = \frac{\|x\|^2}{t}$

$\Rightarrow$ Seek solutions of the form $u(t,x) = v(\sigma)$, $\sigma(t,x) = \frac{\|x\|^2}{t}$.

u solves the original PDE $\Leftrightarrow$ v solves an ODE. Calculate this ODE + find that its solution is

$$v(\sigma) = c\int_0^{\sigma} e^{-\frac{1}{4}w}\, w^{-1/2}\, dw + d.$$

Unwieldy. But notice that if u is a sol, so is $\frac{\partial u}{\partial x}$:

$$u(x,t) = c\,\frac{1}{t^{1/2}}\, e^{-x^2/4t} \qquad (\text{if } x \in \mathbb{R})$$

$\boxed{\text{EX}}$ (Exercise) Use the chain rule to find & solve the ODE's for Δ in $\mathbb{R}^2$, heat in $\mathbb{R}^+ \times \mathbb{R}$.

Another way to find special solutions: via rescaling.

This method sometimes works also for nonlinear equations.

Suppose $u(x,t)$ is a solution. Define $v_\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$. | S
Require that if u is a solution, then so is v_λ. This forces a relationship between α and β. Then, want u to be invariant under this, ie,

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t), \quad \forall \lambda.$$

If this is so, then in particular it will hold for $\lambda = t^{-1}$, so that

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, 1) = t^{-\alpha} u(t^{-\beta} x, 1).$$

So u is really just a function of $t^{-\beta} x$. (If in higher dimensions, try $t^{-\beta}\|x\|$.)

Put $J = t^{-\beta} x$, & reduce to ODE in same manner as above.

$\boxed{EX}$ (Exercise) Find special solution to Burger's equation by this method: $u_t + u u_x = 0$. Also try symmetry method on this equation.

Other methods - separation of variables, method of characteristics, etc.

$\boxed{EX}$ (Exercise) Also solve Burger's eq. by separation of vars & by method of characteristics.

One use of special solutions - as comparisons.

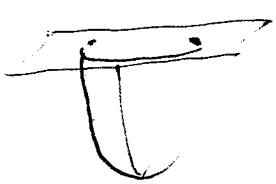
Use these comparison sol's to prove bounds or to prove special properties.

$\boxed{\text{Ex}}$ $Pu = \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u$,

$(a^{ij}) > 0$ (i.e. P elliptic) and $c \leq 0$ in Ω.

(For ex. $P = \Delta$.)

Suppose $Pu \geq Pv$. Geometrically, u has more upward curvature. If $u = v$ on $\partial\Omega$, or if $u \leq v$ on $\partial\Omega$, then the Comparison Theorem states that $u \leq v$ ~~th~~ throughout Ω.



Often can find simple geometries - sphere, plane, etc - that is either sub- or supersolution. Can compare on $\partial\Omega$ if a boundary value is prescribed. Then can compare throughout Ω; get bound.

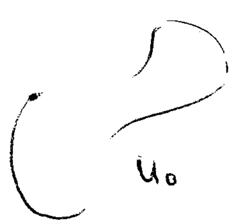
$\boxed{\text{EX}}$ $\begin{cases} \dfrac{\partial u}{\partial t} - \Delta(u^\gamma) = 0 \\ u(x,0) = f(x) \end{cases}$ IVP for Porous Medium.

The sol's to the PDE which are produced by the scale-invariant method above are called Barenblatt sol's. Because explicit, can see directly that they have finite propagation speed:

A solution generally won't be both scale-invariant and satisfy the initial value. $\Rightarrow$ actual solution not known explicitly.

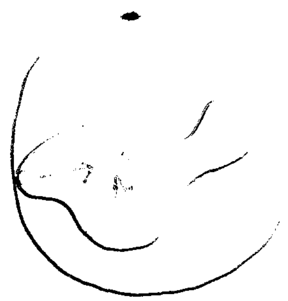
Use comparison to trap actual sol. between 2 Barenblatt sol's at $t=0$. By comparison, it remains between them. Therefore, also has finite propagation speed.

$\boxed{\text{EX}}$ $\dfrac{\partial \vec{P}}{\partial t} = k \cdot \hat{N}$, Curve-shortening flow



Enclose by circle - can compute flow exactly.

Then $u(t,x)$ remains within shrinking circle. Because if not, ~~the~~ there'd be a first moment at which they touch:



Inner curve has greater curvature, hence moves inward faster.

Another use of special solutions — construct other solutions.

Ideally, will be able to construct all other sol's in terms of the special one you found — call them fundamental solution. (Note that we solved only the PDE; we didn't satisfy any ∂ values or initial values.)

$\boxed{\text{Ex}}$ Initial Value Problem for the heat operator in $\mathbb{R}^n$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 \\ u(x,0) = f(x). \end{cases}$$

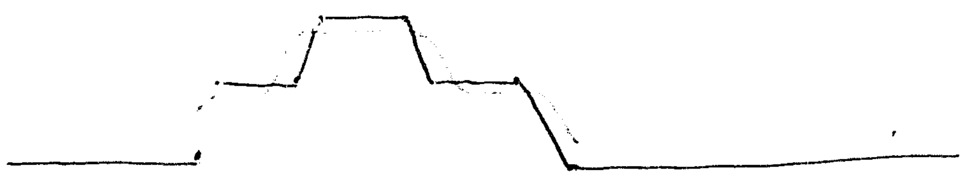
If $f \in C^0(\mathbb{R}^n)$ and bounded, then solution is

$$u(x,t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t} f(y)\, dy.$$

Can read off some important properties.

Infinite propagation speed: Suppose $f(x) \geq 0$, and compactly supported. Then $\forall\, t > 0$, $u(x,t) > 0$.

Smoothing effect: Note that u is given as an improper integral. It converges because $e^{-x} \to 0$ as $|x| \to \infty$ fast enough. $\partial u / \partial x_i$? $x e^{-x} \to 0$, and $x^k e^{-x} \to 0$. $\Rightarrow$ u smooth.



Compare to limited regularity of the solutions to Poisson eq.

$\boxed{\text{Ex}}$ $\Delta u = 0$ or $\Delta u = f$, with either u or its normal deriv. given on $\partial\Omega$.

Recall that by assuming radial symmetry we derived special solution

$$\Gamma(x) = \begin{cases} C_2 \log \|x\|, & n = 2 \\ C_n \|x\|^{2-n}, & n \geq 3. \end{cases}$$

This was in $\mathbb{R}^n$, or at least without reference to a particular domain or boundary values.

Closely related to the fundamental solution Γ but depending on the choice of domain, have a Green's function, $G(x,y) = G(y,x)$. $G(x,y)$ and $\Gamma(x-y)$ have same regularity.

If $u \in C^2(\bar{\Omega})$, using integration by parts + properties of the fundamental solution or Green's fcn, ~~we~~ have the identity

$$u(x) = \int_{\partial\Omega} \left[G(x,y)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial G}{\partial \nu}(x,y) \right] d\sigma_y - \int_{\Omega} G(x,y)\,\Delta u(y)\,dy.$$

Looking at it the other way, to solve $\Delta u = 0$ or $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, substitute these in on the rhs and then ask whether u solves the eq + what regularity it has.

If $\Delta u = 0$ in Ω, u is smooth; first term ok

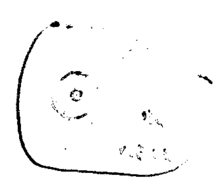
But compare to $\Delta u = f$.

$\|x-y\|$ never zero if $y \in \partial\Omega$, $x \in \Omega$.

$\Delta : C^2(\Omega) \to C^0(\Omega)$, so if $f \in C^0(\Omega)$, might expect sol. u to be C^2. But it isn't.

Write in polars around x. Then near x,

$$G(r) \sim r^{2-n}.$$



second term is $\int_0^1 r^{2-n}\, r^{n-1}\, dr < \infty$.

After one deriv:

$$\frac{\partial u}{\partial x_i} = \cdots + \int_{\Omega} \quad \sim \int_0^1 r^{1-n} \, r^{n-1} \, dr < \infty,$$

after 2 deriv's, $\int_0^1 r^{-n} r^{n-1} dr$: Not integrable.

This is precisely the borderline case. If $f \in C^{\alpha}(\Omega)$, can borrow extra differentiability & get $\int_0^1 r^{-n} r^{\alpha} r^{n-1} dr = \int_0^1 r^{-1+\alpha} dr$.

$$\Rightarrow \quad \Delta \; C^{1,2}(\Omega) \xrightarrow{\Delta} C^{0}(\Omega)$$

$$C^{1,2}(\Omega) \xleftarrow{\text{“}\Delta^{-1}\text{”}} C^{\alpha}(\Omega).$$

In both cases, singularities of the integral kernel determined mapping properties of the solution operators.

$\boxed{\text{Ex}}$ Solutions may lie in weighted spaces: Heat (or other parabolic) equation in bounded domain Ω, where initial value does not satisfy the PDE at $\partial\Omega$.

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④ Fourier Transform, Further Directions.

If $f \in L^1(\mathbb{R}^n)$ (and by completeness can also extend to fcns in $L^2(\mathbb{R}^n)$), define FT:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\,dx,$$

and the IFT:

$$\check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi)\,d\xi.$$

The great utility of FT in PDEs comes from the way it interacts with derivatives.

$\boxed{\text{EX}}$ Take the FT of the deriv of u.

$$\left(\frac{\partial}{\partial x_j} u\right)^{\wedge}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-i\langle x,\xi\rangle} \frac{\partial u}{\partial x_j}\,dx.$$

If $u, u_{x_j} \to 0$ fast enough at ∞, can integrate by parts without introducing any boundary terms (This will be the case if $u \in \mathcal{S}$, the Schwartz space.)

$$= -\frac{1}{(2\pi)^{n/2}} \int \left(\frac{\partial}{\partial x_j} e^{-i\langle x,\xi\rangle}\right) u(x)\,dx$$

$$= \cancel{i\xi_j \int} = i\xi_j \frac{1}{(2\pi)^{n/2}} \int e^{-i\langle x,\xi\rangle} u(x)\,dx = i\xi_j \hat{u}(\xi).$$

Solve PDE's by applying FT to convert into algebraic eq in $\hat{u}$; solve for $\hat{u}$ and IFT back.

$\boxed{\text{Ex}}$ Produce fund. solution for heat eq and for wave eq.
$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$
by taking FT in spatial variables only, to produce an ODE in $\hat{u}(t)$. Solve, and IFT back.

$\boxed{\text{Ex}}$ ~~Laplace operator in~~ $\cancel{\mathbb{R}^?}$ $\mathbb{R}^2$.
$$\cancel{\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)}$$

Write u as IFT of its own FT:
$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_\xi} e^{i\langle x, \xi\rangle} \hat{u}(\xi)\, d\xi,$$
and then apply a differential operator $P(x, D)$.

$\boxed{\text{Ex}}$ Apply Laplace operator to u, in $\mathbb{R}^2$:
$$\underbrace{\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)}_{= P(x, D)} u(x) = \frac{1}{(2\pi)^{n/2}} \int \underbrace{-(\xi_1^2 + \xi_2^2)}_{= P_L(x, \xi)} e^{i\langle x, \xi\rangle} \hat{u}(\xi)\, d\xi.$$

Here we <u>started</u> with diff. operator ~~pt~~ $P(x, D)$, & calculated the polynomial $p(x, \xi)$. But we could start with $p(x, \xi)$ and use this to <u>define</u> the action of an operator

$p(x, D)$ on u. Broaden the class of fcns that are allowed to play the role of $p(x,\xi)$ and you correspondingly broaden the types of operators acting on u — maybe even capturing inverse or solution operators. "Pseudodifferential."

<u>Remark:</u> Solution or inverse operator to a diff. operator wouldn't be a differential operator. If $u(x) \equiv 0$ in U, then all deriv's also zero. But as we saw in the heat equation ex., solution operator can increase support.

$\boxed{\text{EX}}$ Solve Poisson by FT?

$$\Delta u = f \Rightarrow (\Delta u)^\wedge = f^\wedge \Rightarrow -(\xi_1^2 + \dots + \xi_n^2)\hat{u} = \hat{f}$$

$$\Rightarrow \hat{u}(\xi) = \frac{-1}{\xi_1^2 + \dots + \xi_n^2}\hat{f}(\xi).$$

But rhs undefined. To calculate IFT, must integrate over all of $\mathbb{R}^n_\xi$, including $\vec{\xi} = \vec{0}$.

Modify idea: Introduce cut-off fcn. The corresponding pseudodifferential operator can be considered a first approximation to "Δ^{-1}."

References

For Fourier Transform: Folland.

Explicit methods of solution: Evans.

Pseudodifferential Operators: Taylor, Vol. 416 LNM or
more recent book.

Comparison Theorems: Gilbarg + Trudinger (elliptic)
Lieberman (parabolic).

Applications of Comp. Th: Tso, Gauss curvature flow,
Aronson (?), finite propagation speed
for porous medium eq.

Extemporaneous remarks added during talk

1) Significance of singularities of explicit sol's to Laplace equation:

We obtained $\Phi(x) = C_2 \log\|x\|$ and $\Phi(x) = C_n \|x\|^{2-n}$, both of which are singular at $x=0$. This is not a feature artificially introduced by choice of polar coord's. Indeed, the choice of coord's was motivated by the operator itself. We will see that this reflects something fundamental to the operator, namely the regularity properties of solutions.

2) Motivation for allowing more general $p(\xi)$:

In one spatial variable, easily solve the wave equation by factoring. $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u$. In higher dimensions, to generalize this one needs to make sense of $\left(\frac{\partial}{\partial t} - \Delta^{1/2}\right)\left(\frac{\partial}{\partial t} + \Delta^{1/2}\right)$! Likewise, $e^{-t\Delta}$ occurs naturally in the study of the heat eq

Thursday, Aug 28. Thalia Jeffres

1. Differential Equations & Operators : linear & nonlinear.

$\Delta u = 0$, $\Delta u = f$

$Pu = 0$, $Pu = f$

Δ, P

$e^x \not\Rightarrow$ nonlinear

$e^u \Rightarrow$ nonlinear

Examples:

$u_t + u u_x = 0$ Burgers Equation

$u_t - \Delta(u^r) = 0$ $r > 1$.

Example:

Reaction - Diffusion Equation

$u_t - \Delta u = f(u)$

f nonlinear in u.

Semilinear : the highest order part is linear

Example:

Minimal surface Equation:

$$\mathrm{div}\left(\frac{1}{(1+|\nabla u|^2)^{\frac{1}{2}}} \cdot \nabla u\right) = 0 \text{ on } f.$$

Quasilinear equation

Examples:

$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, in a region intersecting

$\{x = 0\} \Rightarrow$ Problem:

$\frac{1}{1+x^2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in a region x unbounded

Example:

Minimal Surface equations.

$$\underbrace{\frac{1}{(1+|\nabla u|^2)^{\frac{1}{2}}}}_{\uparrow} \Delta u + \{\text{lower order terms}\}$$

Could be unbounded even in a bdd region

Example (PM)

$$u_t - \nu u^{\nu-1} \Delta u + \{\text{lower order terms}\}$$

$\boxed{\text{Linearization}}$

$P: \underset{\widehat{\mathcal{H}}}{\mathcal{U}} \to \mathcal{G}$ $\mathcal{H}$ is a function space

$dP_u : \mathcal{H} \to \mathcal{G}$

$$dP_u(v) = \frac{d}{d\sigma}\Big|_{\sigma=0} P(u+\sigma v)$$

Example:

$P = u_t - \Delta u - f(u)$

$$dP_u(v) = \frac{d}{d\sigma}\Big|_{\sigma=0} P(u+\sigma v)$$

$$= v_t - \Delta v - f'(u)\cdot v$$

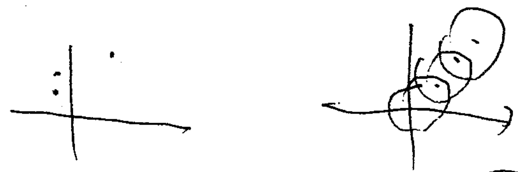
Examples

Exercise: Find linearization for (PM)

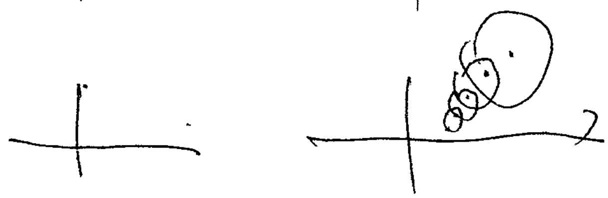
$dP_u(v) = v_t - \nu u^{\nu-1} \Delta v + \{\text{lower order terms}\}$

$P:(\mathfrak{F}, \|\cdot\|_{\mathfrak{F}}) \to (\mathcal{G}, \|\cdot\|_{\mathcal{G}})$

$Pu = 0$



good situation



bad

- need: Control over radii of balls

Nonlinear Techniques

Iterate

Fixed point Thms

Inverse, Implicit Function Thms

Continuity method

~~Compactness Thm~~

Example:

Solve:

$$u'(t) - u(t) - f(u(t)) = 0$$

$$u'(t) - u(t) = f(v(t))$$

$$u = \mathcal{P}(v) = ce^t + \int_0^t e^{t-s} f(v(s))\,ds$$

The original equation is solved by fixed point of $\mathcal{P}$:

Example of finding Uniform estimates. (on u)

$\Delta u(p_0) \leq 0$

at maximum p_0 of u

Suppose: $\Delta u \geq \cdots \geq Au - B$

with $A > 0$

at maximum:

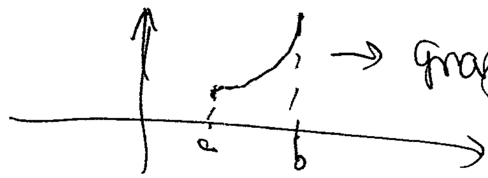
$$0 \geq \Delta u(p_0) \geq A u(p_0) - B$$

$$u \leq u(p_0) \leq \frac{B}{A}$$

Next step: $\quad 1 + |\nabla u|^2$

② Variational Problems

Example: shortest curve



→ graph of f

$$l(f) = \int_a^b \sqrt{1+f'(x)^2}\, dx$$

$g(a) = g(b) = 0$ g also in C^1

$l(f) \le l(f+tg)$ $\forall t$

$$\frac{d}{dt}\Big|_{t=0} l(f+tg) = 0$$

Calculations:

$$\frac{d}{dt}\Big|_{t=0} l(f+tg) = \int_a^b \frac{f_x}{\sqrt{1+f'(x)^2}}\, g_x\, dx$$

$$= -\int_a^b \boxed{\frac{d}{dx}\left(\frac{f_x}{\sqrt{1+f'^2(x)}}\right)} g(x) = 0$$

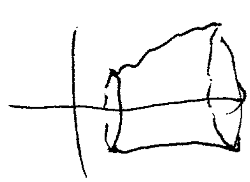
$$\boxed{\frac{d}{dx}\left(\frac{f_x}{\sqrt{1+f'^2(x)}}\right)} = 0$$

Euler–Lagrange Equation

$f_{xx} = 0$

Example:

Exercise:



Surface of revolution

minimize area:

$u \in C^1(\Omega)$, $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{\partial u}{\partial x} \cdot \phi = -\int_\Omega u \frac{\partial \phi}{\partial x} \, dx$$

Weak derivative

$H^k(\Omega) = \{ u$ and up to k weak derivatives are in $L^2(\Omega) \}$

③ Special solutions

$Pu = 0$ in $\mathbb{R}^n$ or $\Omega \subseteq \mathbb{R}^n$

A: a linear $\mathbb{R}^n \to \mathbb{R}^n$ is a symmetry of P if whenever u is a solution

$$v = u \circ A$$

is a solution

Example: $O(n)$ is symmetry of Δ

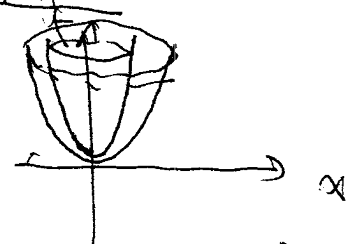
The orbit of $O(n)$ is a sphere which is characterized by radius

$$u(x) = v(r(x))$$

Solves PDE $\Leftrightarrow$ solves ODE

$$u(x) = \begin{cases} C_2 \log \|x\| & n = 2 \\ C_n \|x\|^{2-n} & n > 2 \end{cases}$$

Example: $\partial_t - \Delta$

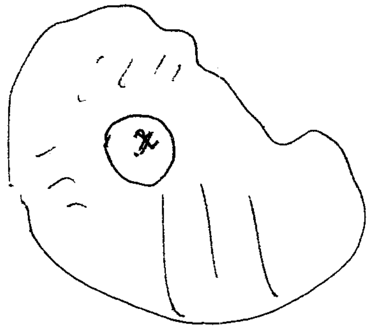


$$r = \frac{\|x\|^2}{t}$$

Seek solutions of r

$$u(x,t) = v(r)$$

$$v(r) = c_1 \int_0 e^{-\frac{1}{4}w} w^{\frac{1}{2}} \, dw + C_2$$



$\Omega = \Omega' \cup$ Ball

$G(x,y) \sim r^{2-n}$

$u = \int_{\partial\Omega} + \int_{\Omega'} + \int_0^\rho r^{2-n}\, r^{n-1} f(y)\, dr\, d\theta$

$\uparrow$ good $\quad \uparrow$ good $\quad \uparrow$ fine

$\frac{\partial u}{\partial x_j} = \cdots + \int_0^\rho r^{1-n}\, r^{n-1} f(y)\, dr\, d\theta$

$\uparrow$ ok

$\frac{\partial^2 u}{\partial x_j \partial x_k} = \cdots + \int_0^\rho r^{-n}\, r^{n-1} f(y)\, dr\, d\theta$

$\uparrow$ problem

$C^2(\Omega) \xrightarrow{\Delta} C^0(\Omega)$

?? $C^2(\Omega) \xleftarrow{\text{"}\Delta^{-1}\text{"}} C^0(\Omega)$ in general NO.!!

$C^2(\Omega) \leftarrow C^\alpha(\Omega)$

$\frac{\partial^2 u}{\partial x_j \partial x_k} = \cdots + \int_0^\rho r^{-n}\, r^{\alpha}\, r^{n-1}\, r^{-\alpha} f(y)\, dr\, d\theta$

$\leq \int_0^\rho r^{-1+\alpha}\, r^{-\alpha} f(y)\, dy$ ← good

④ Fourier Transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} e^{-ix\cdot\xi} f(x)\,dx$$

$$\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,dx$$

Exercise: $\hat{f}$ is Inverse to $\check{f}$

$$\left(\frac{\partial}{\partial x_j} u\right)^{\wedge}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \frac{\partial u}{\partial x_j}\,dx$$

$$= i\xi_j\,\hat{u}(\xi)$$

$\boxed{\text{Ex}}$ Heat:

Wave:

Take Fourier Transform on x only

$$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n_\xi} e^{i\langle x,\xi\rangle}\,\hat{u}(\xi)\,d\xi$$

<u>Example:</u>

$$\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} -(\xi_1^2 + \xi_2^2)\, e^{i\langle x,\xi\rangle}\,\hat{u}(\xi)\,d\xi$$

$$\sqrt{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}\; u = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_\xi} \sqrt{-(\xi_1^2 + \xi_2^2)}\; e^{i\langle \xi,x\rangle}\,\hat{u}(\xi)\,d\xi$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = 0 \Rightarrow \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u = 0$$