The Atiyah-Singer Index Theorem II

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Where we ended last time

Let M be an oriented, compact, even-dim. Riemannian manifold without boundary.

- Let E and F be Hermitian vector bundles on M and let

$$L : C^\infty(M, E) \to C^\infty(M, F)$$

be a "Gauss-Bonnet type operator" (technically called a **Dirac operator**).

- (**Atiyah-Singer Index Theorem, 1963**) L is Fredholm and the following index formula holds:

$$\boxed{\begin{array}{ccc} \operatorname{ind} L & = & \int_M K_{AS}, \\ \textbf{analytical} & & \textbf{geometrical} \end{array}}$$

where K_{AS} is an **explicitly** defined polynomial in the curvatures of M, E, and F.

Questions

- Last lecture we talked about a “poor man’s” Gauss-Bonnet operator, but a . . .

“poor man’s ____” is a cheaper, simpler version of ____.

So, what is the true Gauss-Bonnet operator?

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  So, what is the true Gauss-Bonnet operator?

- What is a Dirac operator?

- What is K_{AS}?

Outline of talk

I. Review of differential operators

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II. Review of the principal symbol and ellipticity

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I. Review of differential operators

II. Review of the principal symbol and ellipticity

III. Dirac operators

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I. Review of differential operators

II. Review of the principal symbol and ellipticity

III. Dirac operators

IV. The true Gauss-Bonnet operator

Outline of talk

I. Review of differential operators

II. Review of the principal symbol and ellipticity

III. Dirac operators

IV. The true Gauss-Bonnet operator

V. The term K_{AS}

I. Review of differential operators

Preview of Part I

- A **(linear) differential operator** is a linear map given by taking linear combinations of partial derivatives and multiplying by smooth functions.

- The poor man's Gauss Bonnet operator is a first order differential operator.

I. Review of differential operators

Ex 1. $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ is the **Laplacian** or **Laplace operator**:

$$L = \Delta = -\partial_x^2 - \partial_y^2;$$

$$\Delta f = -\partial_x^2 f - \partial_y^2 f.$$

Δ is an example of a **second order operator**.

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Δ is an example of a **second order operator**.

Ex 2. $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$

$$L = -\partial_x^2 - \partial_y^2 + 5\partial_x - x^2\partial_y + 10e^{-x-y}.$$

Another second order operator.

I. Review of differential operators

Ex 3. The **Cauchy-Riemann operator** is the operator $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ defined by

$$D_{CR} = \partial_x + i\partial_y.$$

D_{CR} is an example of a **first order operator**. This operator is the fundamental operator of complex analysis!

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D_{CR} is an example of a **first order operator**. This operator is the fundamental operator of complex analysis!

Ex 4. Another first order operator is $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ defined by

$$L = \partial_x + i\partial_y + 2\sin(x^2 + y^2).$$

I. Review of differential operators

Ex 5. Recall that the **poor man's Gauss-Bonnet operator** is the operator

$$L_{GB} : C^\infty(M, TM) \to C^\infty(M, \mathbb{R}^2)$$

defined by

$$L_{GB}(v) = (-\mathrm{curl}\, v \;,\; \mathrm{div}\, v)$$

where v is a vector field on M.

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$$L_{GB}(v) = (-\mathrm{curl}\, v\ ,\ \mathrm{div}\, v)$$

where v is a vector field on M.

Let $M = \mathbb{R}^2$. Given a vector field $v = f\vec{\imath} + g\vec{\jmath}$ on $\mathbb{R}^2$,

$$\mathrm{curl}\, v = (\partial_x g - \partial_y f)\vec{k}$$

$$\mathrm{div}\, v = \partial_x f + \partial_y g.$$

I. Review of differential operators

Therefore,

$$L_{GB}(f\vec{\imath}+g\vec{\jmath}) = (\partial_y f - \partial_x g,\ \partial_x f + \partial_y g).$$

We can also write L_{GB} as a matrix:

$$\boxed{L_{GB}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}.}$$

Therefore, the poor man's Gauss-Bonnet operator is a first order differential operator.

In general, a differential operator L is of **m-th order** if each term of L involves at most m differentiations.

I. Review of differential operators

Summary of Part I

- A **differential operator** is a linear map given by taking linear combinations of partial derivatives and multiplying by smooth functions.

- The Laplacian is a second order differential operator

- The Cauchy-Riemann operator and the poor man's Gauss Bonnet operator are first order differential operators.

II. Principal symbol and ellipticity

Preview of Part II

- Principal = first or of highest importance, rank, worth.
- The principal symbol of a differential operator is a (matrix of) *polynomials* determined by the "most important" part of the operator.
- A differential operator is elliptic if its principal symbol is invertible.

II. Principal symbol and ellipticity

- Principal symbol.

Ex 2 con't: Consider the operator:

$$L = -\partial_x^2 - \partial_y^2 + 5\partial_x - x^2\partial_y + 10e^{-x-y}.$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 2 con't: Consider the operator:

$$L = -\partial_x^2 - \partial_y^2 + 5\partial_x - x^2\partial_y + 10e^{-x-y}.$$

The principal symbol of L is

$$\begin{aligned}\sigma(L)(\xi_1, \xi_2) &= -(i\xi_1)^2 - (i\xi_2)^2 \\ &= \xi_1^2 + \xi_2^2 \\ &= |\xi|^2 \ \left(\text{where } \xi = (\xi_1, \xi_2)\right) \\ &= \text{squared length of } \xi.\end{aligned}$$

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&= \xi_1^2 + \xi_2^2 \\
&= |\xi|^2 \ \text{(where } \xi = (\xi_1, \xi_2)) \\
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Ex 4 con't: Consider the operator

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The principal symbol of L is

$$\begin{aligned} \sigma(L)(\xi_1, \xi_2) &= i\xi_1 + i(i\xi_2) \\ &= i\xi_1 - \xi_2. \end{aligned}$$

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Ex 3 con't: For the Cauchy-Riemann operator $D_{CR} = \partial_x + i\partial_y$, we have

$$\sigma(D_{CR})(\xi_1, \xi_2) = i\xi_1 - \xi_2.$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 5 con't: Consider the poor man's Gauss-Bonnet operator (written as a matrix)

$$L_{GB} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix}.$$

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$$L_{GB} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix}.$$

The principal symbol of L is

$$\sigma(L_{GB})(\xi_1, \xi_2) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}.$$

II. Principal symbol and ellipticity

- Principal symbol.

Let L be an m-th differential operator and let $x_1, x_2, \ldots, x_n$ be the variables it differentiates with respect to.

In the terms of L containing m partial derivatives, replace

$$\partial_{x_1} \text{ by } i\xi_1 \,, \qquad \partial_{x_2} \text{ by } i\xi_2 \,, \qquad \ldots \,, \qquad \partial_{x_n} \text{ by } i\xi_n.$$

The resulting function of the real variables $\xi_1, \ldots, \xi_n$ is called the **principal symbol** of L:

$$\sigma(L)(\xi_1, \ldots, \xi_n) \quad \text{or} \quad \sigma(L)(\xi),$$

where $\xi = (\xi_1, \ldots, \xi_n)$.

II. Principal symbol and ellipticity

- Ellipticity.

Recall that

$$\sigma(\Delta)(\xi) = |\xi|^2.$$

For $\xi \neq 0$; that is, $\xi = (\xi_1, \xi_2) \neq (0, 0)$,

$$\sigma(\Delta)(\xi)^{-1}$$

is defined.

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$$\sigma(\Delta)(\xi)^{-1}$$

is defined.

Similarly, for $\xi \neq 0$

$$\sigma(D_{CR})(\xi) = i\xi_1 - \xi_2$$

is invertible.

II. Principal symbol and ellipticity

- Ellipticity.

The poor man's Gauss-Bonnet operator,

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix},$$

also has the same property:

For $\xi \neq 0$, $\sigma(L_{GB})(\xi)$ is an invertible matrix.

(Notice that $\det \sigma(L_{GB})(\xi) = -\xi_2^2 - \xi_1^2 = -|\xi|^2$.)

II. Principal symbol and ellipticity

- Ellipticity.

A differential operator L is **elliptic** if for $\xi \neq 0$, the principal symbol $\sigma(L)(\xi)$ is invertible.

Thus, Δ, D_{CR}, and L_{GB} are elliptic.

II. Principal symbol and ellipticity

- Ellipticity.

A differential operator L is **elliptic** if for $\xi \neq 0$, the principal symbol $\sigma(L)(\xi)$ is invertible.

Thus, Δ, D_{CR}, and L_{GB} are elliptic.

Most operators are not elliptic! E.g.

$$L = \partial_x^2 - \partial_y + 10.$$

We have $\sigma(L)(\xi_1, \xi_2) = -(i\xi_1)^2 = \xi_1^2$. Then $\xi = (0, 1) \neq 0$, but

$$\sigma(L)(\xi) = 0 \text{ is not invertible.}$$

II. Principal symbol and ellipticity

Summary of Part II

- Examples: $\sigma(\Delta)(\xi) = |\xi|^2$, $\sigma(D_{CR})(\xi) = i\xi_1 - \xi_2$, and

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}$$

- Significance: Laplacians involve geometry. What is the significance of the last two examples?

- The three operators above are elliptic.

- Can also define differential operators, principal symbols, and ellipticity when manifolds and vector bundles are involved.

III. Dirac operators

Preview of Part III

- Recall the Laplacian is a second order operator such that

$$\sigma(\Delta)(\xi) = |\xi|^2.$$

Thus, Δ captures geometry.

- A Dirac operator is a first order operator whose principal symbol "squared" is $|\xi|^2$.

III. Dirac operators

Ex. For $D_{CR} = \partial_x + i\partial_y$, we have

$$\sigma(D_{CR})(\xi) = i\xi_1 - \xi_2,$$

so

$$\begin{aligned} \overline{\sigma(D_{CR})(\xi)}\,\sigma(D_{CR})(\xi) &= \overline{(i\xi_1 - \xi_2)}(i\xi_1 - \xi_2) \\ &= (-i\xi_1 - \xi_2)(i\xi_1 - \xi_2) \\ &= \xi_1^2 + \xi_2^2 \\ &= |\xi|^2. \end{aligned}$$

Hence we can obtain lengths (geometry) by conjugating and then multiplying!

III. Dirac operators

Ex. For the poor man's Gauss-Bonnet operator, we have

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}.$$

III. Dirac operators

Ex. For the poor man's Gauss-Bonnet operator, we have

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \sigma(L_{GB})(\xi)^* \, \sigma(L_{GB})(\xi) &= \begin{pmatrix} -i\xi_2 & -i\xi_1 \\ i\xi_1 & -i\xi_2 \end{pmatrix} \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} \\ &= \begin{pmatrix} \xi_1^2 + \xi_2^2 & 0 \\ 0 & \xi_1^2 + \xi_2^2 \end{pmatrix} \\ &= |\xi|^2. \end{aligned}$$

III. Dirac operators

Definition: A first order differential operator L is called a **(generalized) Dirac operator** if L is elliptic and

$$\sigma(L)(\xi)^* \sigma(L)(\xi) = |\xi|^2.$$

Therefore, D_{CR} and L_{GB} are Dirac operators.

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Definition: A first order differential operator L is called a **(generalized) Dirac operator** if L is elliptic and

$$\sigma(L)(\xi)^*\sigma(L)(\xi) = |\xi|^2.$$

Therefore, D_{CR} and L_{GB} are Dirac operators.

- Recall

$$\sigma(\Delta)(\xi) = |\xi|^2.$$

Thus, we can think of a Dirac operator as an operator such that when you square it (really, the principal symbol), you get the (principal symbol of the) Laplacian.

Hence, a Dirac operator is a type of “square root” of a Laplacian.

III. Dirac operators

- Dirac operators can be defined when Riemannian manifolds and Hermitian vector bundles are involved: as a first order differential operator L that is elliptic and

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III. Dirac operators

- Dirac operators can be defined when Riemannian manifolds and Hermitian vector bundles are involved: as a first order differential operator L that is elliptic and

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- *Now we understand the hypothesis of Atiyah-Singer!*

"Let E and F be Hermitian vector bundles on M and let

$$L : C^\infty(M, E) \to C^\infty(M, F)$$

be a "Gauss-Bonnet type operator" (technically called a **Dirac operator**)."

III. Dirac operators

Summary of Part III

- A Dirac operator is a first order differential operator whose principal symbol "squared" is the symbol of the Laplacian.

- Like Laplacians, Dirac operators capture the geometry of the manifold.

- Advantage of Dirac operators: They are first order instead of second order. (Hence are simpler "in principle.")

IV. The true Gauss-Bonnet operator

Preview of Part IV

- Differential forms are objects you integrate (in line and area integrals).
- The exterior derivative d is just the gradient and curl "all-in-one".
- The Gauss-Bonnet operator is $D_{GB} = d + d^*$.

IV. The true Gauss-Bonnet operator

- Differential forms. **(Focus on $\mathbb{R}^2$.)**

$$C^\infty(\mathbb{R}^2, \Lambda^0) = C^\infty(\mathbb{R}^2) = \text{0-forms}$$

$$C^\infty(\mathbb{R}^2, \Lambda^1) = \text{1-forms} \quad f\,dx + g\,dy$$

$$C^\infty(\mathbb{R}^2, \Lambda^2) = \text{2-forms} \quad f\,dx \wedge dy$$

There are no 3-forms on $\mathbb{R}^2$.

IV. The true Gauss-Bonnet operator

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There are no 3-forms on $\mathbb{R}^2$.

Think of

$$dx \longleftrightarrow \vec{\imath}\ , \quad dy \longleftrightarrow \vec{\jmath}\ , \quad dx \wedge dy \longleftrightarrow \vec{k}.$$

Remark: 1-forms are objects usually found in line integrals and 2-forms are found in area integrals.

IV. The true Gauss-Bonnet operator

- The wedge.

The “wedge” $\wedge$ has the defining “cross product” property

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

for any 1-forms α and β. $\Big($ cf. $v \times w = -w \times v\,.\Big)$

IV. The true Gauss-Bonnet operator

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for any 1-forms α and β. $\left(\text{cf. } v \times w = -w \times v \ . \right)$

Ex. $dx \wedge dy = -dy \wedge dx.$ $\left(\text{cf. } \vec{\imath} \times \vec{\jmath} = -\vec{\jmath} \times \vec{\imath}\right).$

IV. The true Gauss-Bonnet operator

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Ex. $$dx \wedge dy = -dy \wedge dx. \qquad \left(\text{cf. } \vec{\imath} \times \vec{\jmath} = -\vec{\jmath} \times \vec{\imath}\right).$$

Ex.

$$\alpha \wedge \alpha = -\alpha \wedge \alpha.$$

Therefore, $\alpha \wedge \alpha = 0$. In particular,

$$dx \wedge dx = 0 \quad \text{and} \quad dy \wedge dy = 0. \qquad \left(\text{cf. } \vec{\imath} \times \vec{\imath} = 0\right).$$

IV. The true Gauss-Bonnet operator

The **exterior derivative**

$$d : C^\infty(\mathbb{R}^2, \Lambda^k) \to C^\infty(\mathbb{R}^2, \Lambda^{k+1})$$

is the differential operator

$$d = \partial_x \, dx + \partial_y \, dy$$

acting componentwise. $\left(\text{cf. } \nabla = \partial_x \, \vec{\imath} + \partial_y \, \vec{\jmath}.\right)$

IV. The true Gauss-Bonnet operator

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Note: d really consists of three maps

$$C^\infty(\mathbb{R}^2, \Lambda^0) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^2) \xrightarrow{d} 0.$$

($d = 0$ on 2-forms since there are no 3-forms.)

IV. The true Gauss-Bonnet operator

0-forms:

$$d : C^\infty(\mathbb{R}^2, \Lambda^0) \to C^\infty(\mathbb{R}^2, \Lambda^1).$$

For $f \in C^\infty(\mathbb{R}^2, \Lambda^0) = C^\infty(\mathbb{R}^2)$,

$$df = \partial_x f \, dx + \partial_y f \, dy.$$

$\left(\text{cf. } \nabla f = \partial_x f \, \vec{\imath} + \partial_y f \, \vec{\jmath}.\right)$ Thus,

$$d = \text{gradient on 0-forms.}$$

IV. The true Gauss-Bonnet operator

1-forms:

$$d(f\,dx + g\,dy) = df \wedge dx + dg \wedge dy$$

IV. The true Gauss-Bonnet operator

1-forms:

$$
\begin{aligned}
d(f\,dx + g\,dy) &= df \wedge dx + dg \wedge dy \\
&= (\partial_x f\,dx + \partial_y f\,dy) \wedge dx + (\partial_x g\,dx + \partial_y g\,dy) \wedge dy \\
&= \partial_y f\,dy \wedge dx + \partial_x g\,dx \wedge dy \\
&= -\partial_y f\,dx \wedge dy + \partial_x g\,dx \wedge dy \\
&= (\partial_x g - \partial_y f)\,dx \wedge dy.
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IV. The true Gauss-Bonnet operator

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&= \partial_y f\,dy \wedge dx + \partial_x g\,dx \wedge dy \\
&= -\partial_y f\,dx \wedge dy + \partial_x g\,dx \wedge dy \\
&= (\partial_x g - \partial_y f)\,dx \wedge dy.
\end{aligned}
$$

$\Big($cf. $\mathrm{curl}(f\,\vec{\imath} + g\,\vec{\jmath}) = (\partial_x g - \partial_y f)\,\vec{k}.\Big)$ Thus,

$d =$ curl on 1-forms.

IV. The true Gauss-Bonnet operator

- Adjoint: If L is an $m \times n$ matrix, we have

$$L : \mathbb{R}^n \to \mathbb{R}^m.$$

The adjoint (conjugate transpose) L^* is an $n \times m$ matrix, so

$$L^* : \mathbb{R}^m \to \mathbb{R}^n.$$

Taking the adjoint switches the domain and codomain.

IV. The true Gauss-Bonnet operator

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The adjoint (conjugate transpose) L^* is an $n \times m$ matrix, so

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Taking the adjoint switches the domain and codomain.

Recall

$$C^\infty(\mathbb{R}^2, \Lambda^0) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^2).$$

There is an adjoint

$$C^\infty(\mathbb{R}^2, \Lambda^2) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^0).$$

IV. The true Gauss-Bonnet operator

- What is d^*?

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- What is d^*?

2-forms: $d^* : C^\infty(\mathbb{R}^2, \Lambda^2) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^1)$:

$$d^*(f\,dx \wedge dy) = \partial_y f\,dx - \partial_x f\,dy.$$

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$$d^*(f\, dx \wedge dy) = \partial_y f\, dx - \partial_x f\, dy.$$

Note:

$$\operatorname{curl}(f\, \vec{k}) = \partial_y f\, \vec{\imath} - \partial_x f\, \vec{\jmath}.$$

Therefore,

$$d^* = \operatorname{curl} \text{ on 2-forms.}$$

IV. The true Gauss-Bonnet operator

- What is d^*?

1-forms: $d^* : C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^0)$:

$$d^*(f\,dx + g\,dy) = -(\partial_x f + \partial_y g).$$

IV. The true Gauss-Bonnet operator

- What is d^*?

1-forms: $d^* : C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^0)$:

$$d^*(f\,dx + g\,dy) = -(\partial_x f + \partial_y g).$$

Note:

$$\operatorname{div}(f\,\vec{\imath} + g\,\vec{\jmath}) = \partial_x f + \partial_y g.$$

Therefore,

$$d^* = -\operatorname{div} \text{ on 1-forms.}$$

IV. The true Gauss-Bonnet operator

- The true Gauss-Bonnet operator. Recall

$$d : C^\infty(\mathbb{R}^2, \Lambda^k) \longrightarrow C^\infty(\mathbb{R}^2, \Lambda^{k+1}).$$

$$d^* : C^\infty(\mathbb{R}^2, \Lambda^{k+1}) \longrightarrow C^\infty(\mathbb{R}^2, \Lambda^k).$$

IV. The true Gauss-Bonnet operator

- The true Gauss-Bonnet operator. Recall

$$d : C^\infty(\mathbb{R}^2, \Lambda^k) \longrightarrow C^\infty(\mathbb{R}^2, \Lambda^{k+1}).$$

$$d^* : C^\infty(\mathbb{R}^2, \Lambda^{k+1}) \longrightarrow C^\infty(\mathbb{R}^2, \Lambda^k).$$

- $C^\infty(\mathbb{R}^2, \Lambda^{ev}) =$ linear combination of 0 and 2 forms
- $C^\infty(\mathbb{R}^2, \Lambda^{odd}) = C^\infty(\mathbb{R}^2, \Lambda^1) =$ 1-forms. Then,

$$D_{GB} = d + d^* : C^\infty(\mathbb{R}^2, \Lambda^{ev}) \to C^\infty(\mathbb{R}^2, \Lambda^{odd})$$

is called the **THE Gauss-Bonnet operator**.

IV. The true Gauss-Bonnet operator

- Exercises:

1) Check that D_{GB} is a Dirac operator.

2) How is the poor man's Gauss-Bonnet operator related to d and d^*?

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- Exercises:

1) Check that D_{GB} is a Dirac operator.
2) How is the poor man's Gauss-Bonnet operator related to d and d^*?

- Given any Riemannian manifold M, we can define differential forms, d, and d^*. Let

$$C^\infty(M, \Lambda^{ev}) = \text{ even forms } , \; C^\infty(M, \Lambda^{odd}) = \text{odd forms.}$$

Then,

$$D_{GB} = d + d^* : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd})$$

is called **THE Gauss-Bonnet operator**. D_{GB} is a Dirac operator.

IV. The true Gauss-Bonnet operator

Notes:

- The operator

$$\Delta := (d + d^*)^2$$

is called the **Laplacian** or **Laplace operator**.

- By definition, $d + d^*$ is a square root of the Laplacian.

- $d + d^*$ and Δ are important in "Hodge theory," a subject which relates the kernels and cokernels of these operators to the topology of the manifold. In particular,

$$\text{ind}\, D_{GB} = \chi(M).$$

IV. The true Gauss-Bonnet operator

Summary of Part IV

- Differential forms are objects you integrate.

- The exterior derivative d is the gradient and curl "all-in-one".

- The adjoint d^* is the curl and divergence "all-in-one".

- The Gauss-Bonnet operator is the Dirac operator

$$D_{GB} = d + d^* : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd}).$$

V. The integrand K_{AS}

Preview of Part V

- (One of the) most beautiful formulas in the world:

$$\operatorname{ind} L = \frac{1}{(4\pi i)^m}\int_M \sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)}\operatorname{STr}\left(e^{K_E+K_F+\frac{1}{4}K}\right).$$

V. The integrand K_{AS}

Data:

- Let M be an oriented, compact, even-dim. Riemannian manifold and let E and F be Hermitian vector bundles on M.
- Let K = curvature of M, K_E = curvature of E, K_F = curvature of F.
- Let $L : C^\infty(M, E) \to C^\infty(M, F)$ be a Dirac operator.

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Atiyah-Singer: L is Fredholm and

$$\operatorname{ind} L = \int_M K_{AS},$$

where K_{AS} is an **explicitly** defined polynomial in K, K_E, K_F.

V. The integrand K_{AS}

- $\hat{A}$-genus of M:

$$\sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)}.$$

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$$\mathrm{STr}\left(e^{K_E+K_F+\frac{1}{4}K}\right).$$

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- $K_{AS} = \frac{1}{(4\pi i)^m}\sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)}\mathrm{STr}\left(e^{K_E+K_F+\frac{1}{4}K}\right),$

where $\dim M = 2m$.

V. The integrand K_{AS}

The **Atiyah-Singer theorem** in all its glory: Given

- An oriented, compact, even dimensional (say $2m$) Riemannian manifold M
- Hermitian vector bundles E and F over M
- Dirac operator $L : C^\infty(M, E) \to C^\infty(M, F)$.

Then,

$$\boxed{\operatorname{ind} L = \frac{1}{(4\pi i)^m} \int_M \sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)} \operatorname{STr}\left(e^{K_E + K_F + \frac{1}{4}K}\right).}$$

V. The integrand K_{AS}

Ex. Consider the Gauss-Bonnet operator:

$$D_{GB} : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd}).$$

Recall that (via "Hodge theory") ind $D_{GB} = \chi(M)$.

One can work out that

$$K_{AS} = \frac{1}{(2\pi)^m} \frac{(-K)^m}{m!} = \frac{1}{(2\pi)^m} \mathrm{Pf}(-K),$$

where $\mathrm{Pf}(-K)$ is called the Pfaffian of M.

$\therefore$ **Gauss-Bonnet-Chern theorem:**

$$\boxed{\chi(M) = \frac{1}{(2\pi)^m} \int_M \mathrm{Pf}(-K).}$$

Summary of Talk

Question: What is the true Gauss-Bonnet operator?

Answer: The operator $d + d^*$ acting on even forms. d is the exterior derivative (= gradient and curl "all-in-one") and d^* is the adjoint of d.

Question: What is a Dirac operator?

Answer: Basically a "square root" of a Laplacian.

Question: What is K_{AS}?

Answer: So beautiful, the AS thm. has to be repeated:

$$\boxed{\operatorname{ind} L = \frac{1}{(4\pi i)^m} \int_M \sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)} \operatorname{STr}\left(e^{K_E + K_F + \frac{1}{4}K}\right).}$$

WAIT!

Isn't this supposed to be a conference on SINGULAR analysis?

A-S on singular manifolds

We know that for a *smooth* manifold M without boundary,

$$\operatorname{ind} L = \int_M K_{AS}.$$

What about SINGULAR manifolds like

What is the A-S theorem for such manifolds?

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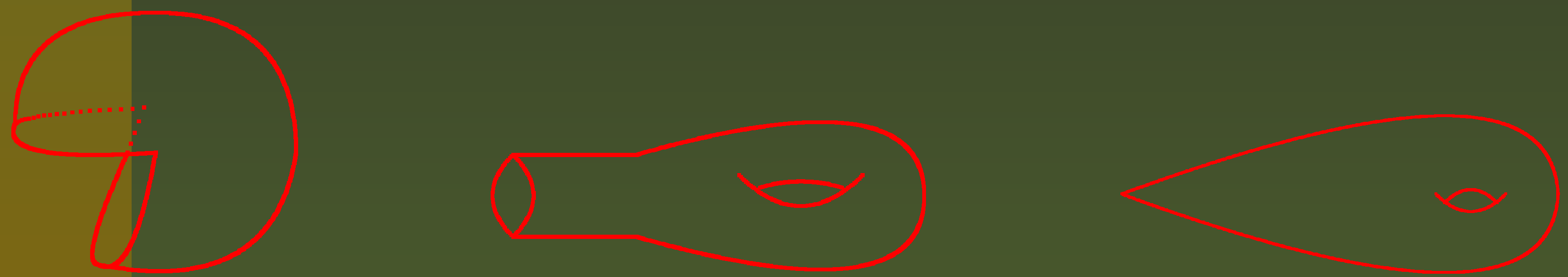

What is the A-S theorem for such manifolds?

Answer: Next week!

An advertisement

We'll talk about

1) Index theorems on singular manifolds.

2) The proof of the A-S theorem: pseudodifferential operators and the heat kernel.

3) The proof of the A-S theorem for a singular manifold: "Exotic" pseudodifferential operators and the heat kernel.

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