Introduction to the Hodge Theorem

Like the index theorem, the Hodge theorem equates two numbers associated to a smooth compact manifold M, one of which has analytical meaning and one of which has topological meaning:

Hodge Thm If M is a compact smooth manifold with metric g then

$$\underbrace{\mathcal{H}^k(M,g)}_{\text{space of k-degree harmonic forms}} \cong \underbrace{H^k(M)}_{\text{k}^{\text{th}}\text{ de Rham cohomology of forms on } M} .$$

Notice both of those spaces are spaces of k-forms, so recall on $\mathbb{R}^2$ we have the sequence of maps

$$\begin{tikzcd}
 & C^\infty(\mathbb{R}^2,\Lambda^0) \arrow[d, equal] & & \\
0 \arrow[r, "d"] & C^\infty(\mathbb{R}^2) \arrow[r, "d"] \arrow[d, equal] & C^\infty(\mathbb{R}^2,\Lambda^1) \arrow[r, "d"] \arrow[d, equal] & C^\infty(\mathbb{R}^2,\Lambda^2) \arrow[r, "d"] \arrow[d, equal] & 0 \\
 & \text{0-forms } f(x,y) & \text{1-forms } f(x,y)\,dx + g(x,y)\,dy & \text{2-forms } f(x,y)\,dx\wedge dy &
\end{tikzcd}$$

where

$d(c) = 0$ is trivial)

$df = (\partial_x f)dx + (\partial_y f)dy$ is like the gradient

$d(f dx + g dy) = (\partial_x g - \partial_y f) dx\wedge dy$ is like the curl

$d(f\, dx\wedge dy) = 0$ is the zero map)

DeRham Cohomology

Notice the following

$$\begin{aligned} d^2 f &= d\left(\partial_x f\, dx + \partial_y f\, dy\right) \\ &= (\partial_x\partial_y f - \partial_y\partial_x f)\, dx\wedge dy = 0 \end{aligned}$$

and $d^2(f dx + g dy) = d(\partial_x g - \partial_y f)dx\wedge dy) = 0$

So $d^2 = 0$. Thus if $\alpha = d\beta$ then $d\alpha = 0$.

$$\overset{\text{"exact forms"}}{\{\text{forms } \alpha \text{ which } = d\beta\}} \subset \overset{\text{"closed forms"}}{\{\text{forms } \alpha \text{ with } d\alpha = 0\}}$$

Further, both of these sets are vector spaces. Thus we can form the quotient space for each k

$$\frac{\{\alpha \in C^\infty(\mathbb{R}^2, \Lambda^k) \mid d\alpha = 0\}}{\{\alpha \in C^\infty(\mathbb{R}^2, \Lambda^k) \mid \alpha = d\beta\}} \doteq H^k(\mathbb{R}^2)$$

This is the <u>k^{th} de Rham cohomology space on $\mathbb{R}^2$</u>

This same setup extends to any smooth manifold, M. If $\dim M = n$, then there are k-forms for $k = 0, 1, 2, \dots, n$:

$$0 \longrightarrow C^\infty(M) = C^\infty(M, \Lambda^1) \xrightarrow{d} C^\infty(M, \Lambda^2) \xrightarrow{d} \cdots \xrightarrow{d} C^\infty(M, \Lambda^n) \rightarrow 0$$

and $d^2 = 0$, so for each k, we can form the quotient space

$$H^k(M) = \frac{\{\alpha \in C^\infty(M, \Lambda^k) \mid d\alpha = 0\}}{\{\alpha \in C^\infty(M, \Lambda^k) \mid \alpha = d\beta\}} = \frac{\text{"closed k-forms"}}{\text{"exact k-forms"}}$$

= k^{th} de Rham cohomology of M.

<u>Ex</u> $M = S^1$

$$0 \longrightarrow C^\infty(S^1) \xrightarrow{d} C^\infty(S^1, \Lambda^1) \xrightarrow{d} 0$$

$$H^0(S^1 = \frac{\{f(\theta) \mid f'(\theta) = 0\}}{\{f(\theta) \mid f(\theta) = 0\}} = \frac{\text{constant functions}}{\{0\}} \cong \mathbb{R}$$

$\{g(\theta)d\theta \in C^\infty(S^1, \Lambda^1)\}$ = <u>all 1-forms</u>

When is $g(\theta)\,d\theta = f'(\theta)\,d\theta$? When

$$f(\theta) = \int_0^{\theta} g(t)\,dt \text{ is smooth.}$$

For this to be true, we need $f(0) = f(2\pi)$, so we need

$$\int_0^{2\pi} g(t)\,dt = 0.$$

But if $g(\theta)\,d\theta \in C^\infty(S^1, \Lambda^1)$ then

$$\tilde{g}(\theta)\,d\theta = \left[g(\theta) - \int_0^{2\pi} g(t)\,dt \right] d\theta \text{ satisfies } \int_0^{2\pi} \tilde{g}(t\,dt = 0,$$

so $\left[g(\theta) - \int_0^{2\pi} g(t)\,dt \right] d\theta = f'(\theta)\,d\theta$ for a smooth f.

so $g(\theta)\,d\theta = C\,d\theta - d(f)$ for $C = \int_0^{2\pi} g(t)\,dt \in \mathbb{R}$.

Thus in the quotient by forms $\{df\}$, only $C\,d\theta$ survives, ie, only constants.

So $H^1(S^1) \cong \mathbb{R}$

So to summarize:

$$H^0(S^1) \cong \mathbb{R} \cong H^1(S^1)$$

Harmonic forms

$$\mathcal{H}^k(M,g) = \{ \alpha \in C^\infty(M,\Lambda^k) \mid (d+d^*)\alpha = 0 \}$$

Recall d^* is the adjoint of d and maps the opposite way from d along the sequence:

$$\begin{tikzcd} 0 & C^\infty(M,\Lambda^0) \arrow[l, "d^*"'] & C^\infty(M,\Lambda^1) \arrow[l, "d^*"'] & \cdots \arrow[l, "d^*"'] & C^\infty(M,\Lambda^n) \arrow[l, "d^*"'] & 0 \arrow[l] \end{tikzcd}$$

It is worth mentioning here that this adjoint is with respect to the L^2 pairing on forms:

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta$$

where $* : C^\infty(M,\Lambda^k) \longrightarrow C^\infty(M,\Lambda^{n-k})$ is a map depending on g. This makes the LHS of the Hodge theorem geometric or analytic, whereas the right side is topological.

Note if $\alpha \in C^\infty(M,\Lambda^k)$ then $d\alpha \in C^\infty(M,\Lambda^{k+1})$
$d^*\alpha \in C^\infty(M,\Lambda^{k-1})$

So there can be no cancellation. Thus

$$\mathcal{H}^k(M,g) = \{ \alpha \in C^\infty(M,\Lambda^k) \mid d\alpha = d^*\alpha = 0 \}.$$

Ex: If $M = S^1$ and assume $g =$ round metric

$$\begin{tikzcd} 0 & C^\infty(S^1) \arrow[l] & C^\infty(S^1,\Lambda^1) \arrow[l, "d^*"'] & 0 \arrow[l] \\ & -h'(\theta) & h(\theta)d\theta \arrow[l] & \end{tikzcd}$$

so $\mathcal{H}^0(S^1,g) = \{ f(\theta) \mid f'(\theta) = 0 \} = \{\text{constants}\} \cong \mathbb{R} \cong H^0(S^1)$

noncompact case

Now consider what happens if we do the same thing on a noncompact manifold, $\mathbb{R}^1$

$\underline{\mathbb{R}}$

$$\begin{tikzcd} 0 \arrow[r] & C^\infty(\mathbb{R}) \arrow[r, "d"] & C^\infty(\mathbb{R},\Lambda^1) \arrow[r] & 0 \\ & f \arrow[r, mapsto] & f'(r)\,dr & \\ & -g'(r) & g(r)\,dr \arrow[l, mapsto, "d^*"'] & \end{tikzcd}$$

$\mathcal{H}^0(\mathbb{R})$ = constant functions $\cong \mathbb{R} \cong$ constant forms $= \mathcal{H}^1(\mathbb{R})$

$$H^0(\mathbb{R}) = \frac{\{\text{constant fns}\}}{\{0\}} \cong \mathbb{R}$$

but since every smooth function on $\mathbb{R}$ is the derivative of its integral, $H^1(\mathbb{R}) = \{0\}$

Thus $\mathcal{H}^1(\mathbb{R}) \not\cong H^1(\mathbb{R})$.

At this point we can take a step back and think if we've gone wrong somewhere. In fact we're been a bit cavalier with our domains.

d^* is the Hilbert space adjoint of d with respect to the pairing on L^2, so really we should be playing this game using L^2 forms, not smooth forms. (This does not change the theorem in the compact case)

So consider

$$\begin{tikzcd} 0 \arrow[r] & L^2(\mathbb{R}) \arrow[r, "d"] & L^2(\mathbb{R},\Lambda^1) \arrow[r] \arrow[l, bend left, "d^*"] & 0 \end{tikzcd}$$

Now we get

$$\mathcal{H}^0_2(\mathbb{R}) = \text{constant } L^2 \text{ functions} \cong \{0\} \cong \text{constant } L^2 \text{ forms} = \mathcal{H}^1_2(\mathbb{R})$$

~~Now~~ So we have

$$\mathcal{H}^1_2(\mathbb{R}) \cong H^1(\mathbb{R}), \text{ but } \mathcal{H}^0_2(\mathbb{R}) \not\cong H^0(\mathbb{R}).$$

So that hasn't helped

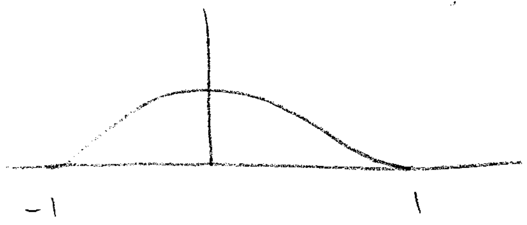
What if we try using $H^\bullet_2(\mathbb{R})$?

Then $$H^0_2(\mathbb{R}) = \frac{\text{constant } L^2 \text{ fns}}{\{0\}} \cong \{0\} \cong \mathcal{H}^0_2(\mathbb{R}) \checkmark$$

What about

$$H^1_2(\mathbb{R}) = \frac{L^2 \text{ 1-forms}}{d(L^2 \text{ fns})} = \frac{\text{kernel } d}{\text{Image } d}$$

Consider the following example:



$g(r)$ where $\int_{-\infty}^{\infty} g(r)\,dr = 1$

and $g \in C^\infty_0(\mathbb{R})$.

Then $$f(r) = \int_{-\infty}^{r} g(\rho)\,d\rho = \begin{cases} 0 & r < -1 \\ \cdots & -1 \le r \le 1 \\ 1 & r > 1 \end{cases} \notin L^2(\mathbb{R}).$$

Further, any other antideriv will = a nonzero constant at $-\infty$ so $g(r)\,dr$ defines a class in $H^1_2(\mathbb{R})$.

However, consider the functions $g_n(r)$:

Then $$f_n(r) = \int_{-\infty}^{r} g_n(\rho)\, d\rho = \begin{cases} 0 & r<-1 \\ \cdots & \\ 0 & r>1+n \end{cases} \in C_0^\infty \subset L^2$$

so $g_n(r)\,dr = d(f_n(r)) = \operatorname{Im} d \quad \forall n$

Further $$\|g-g_n\|_{L^2} = \int_{-\infty}^{\infty} (\text{tail})^2\, d\rho \sim \frac{1}{n} \to 0$$

So $g \in \overline{\operatorname{Im} d}$.

So exact L^2 forms are a dense subspace in closed. This means

$H_2^1(\mathbb{R})$ is ∞ dim'l.

This is related to the fact that the Gauss-Bonnet operator $d+d^*$, which is a Fredholm operator on $L^2(M, \Lambda^\cdot)$ if M is compact, is <u>not</u> generally Fredholm when M is not compact.

There are two ways to deal with this.

One is to consider <u>reduced</u> L^2 cohomology instead of L^2 or regular cohomology. Then

$$\mathcal{H}_2^0(\mathbb{R}) \cong \{0\} \cong H_2^0(\mathbb{R}) \quad \text{and} \quad \mathcal{H}_2^1(\mathbb{R}) \cong \{0\} \cong \overline{H}_2^1(\mathbb{R}) = \frac{\ker d}{\overline{\operatorname{Im} d}}$$

However:
The difficulty here is that reduced L^2 cohomology is not a topological invariant. Nevertheless, it can be made to behave like one sometimes (Mayer-Vietoris sequences, etc), see work of Gilles Carron.

Another approach is to change from L^2 to some weighted space where $d+d^*$ **is** Fredholm. This approach is related to APS, and analytic techniques of the b-calculus.

→ define $L^2(M,\Lambda^i,\varepsilon)$

Thm (APS) If M is a manifold with cylindrical ends then

$$\mathcal{H}_2^i(M,\varepsilon) \cong \frac{\{\sigma\in L^2(M,\Lambda^i,\varepsilon) \mid d\sigma=0\}}{\{d\eta \mid \eta\in L^2(M,\Lambda^i,\varepsilon)\}\cap a^{\varepsilon}L^2(M,\Lambda^i)}$$

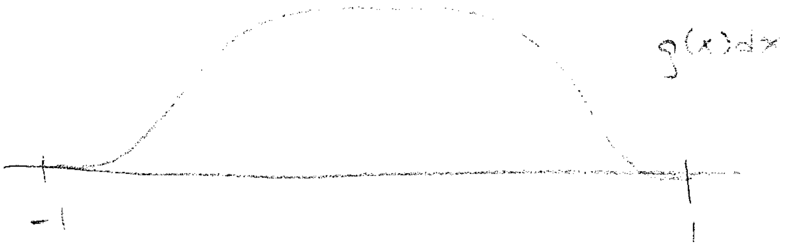
$$\cong \operatorname{Im}\left(H_0^i(M)\longrightarrow H^i(M)\right)$$

↑ de Rham cohomology calculated from C_0^∞ forms

↑ deRham cohomology calculated from C^∞ forms

How does weighting help?

consider our example from before



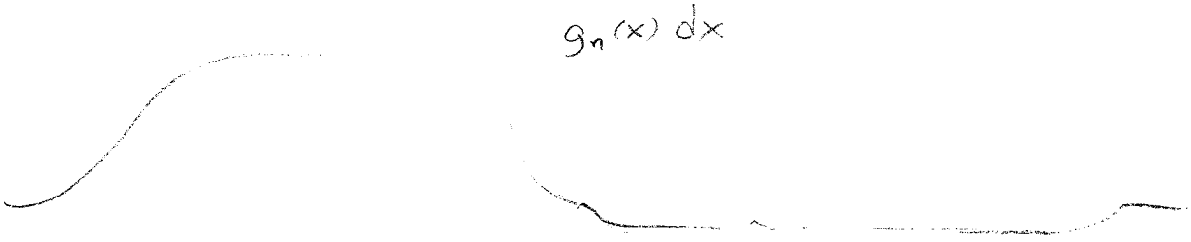
$$f(x) = \int_{-\infty}^{x} g(t)\,dt = \begin{cases} 0 & x < -1 \\ 1 & x > 1 \end{cases}$$

Then $f(x) \in e^{\gamma|x|}L^2$, ie, $f(x) = e^{\gamma|x|} g(x)$, $g \in L^2$

Since $g(x) = e^{-\gamma|x|} f(x) = \begin{cases} 0 & x < -1 \\ e^{-\gamma x} & x > 1 \end{cases}$

so $g(x)\,dx = df$ is okay and $H^1_{2,\gamma}(\mathbb{R}) \cong \{0\} \cong H^1(\mathbb{R})$

<u>or</u> consider $e^{-\gamma|x|}L^2$ then $f(x) \notin e^{-\gamma|x|}L^2$,
but also



$$|g - g_n|_{e^{-\gamma|x|}L^2} = |e^{\gamma|x|}(g - g_n)|_{L^2} \approx \int_1^n e^{2\gamma x}\,dx \to \infty$$

so $g \notin \overline{\operatorname{Im} d(x^{-\gamma|x|}L^2 \to)}$

so $H^1_{2,-\gamma}(\mathbb{R}) \cong \mathbb{R} \cong H^1_{cpt}(\mathbb{R})$

Also, $H^0_{2,\gamma}(\mathbb{R}) \cong \mathbb{R}$ since constants are in $e^{\gamma|x|}L^2$

whereas $H^0_{2,-\gamma}(\mathbb{R}) \cong \{0\}$ since constants are not in $e^{-\gamma|x|}L^2$.

There is a natural map

$$H^i_{2,-\gamma}(\mathbb{R}) \longrightarrow H^i_{2,\gamma}(\mathbb{R}) \qquad \forall\ \gamma>0,\ i$$

and we see

$$\mathrm{Im}\,(H^i_{2,-\gamma}(\mathbb{R}) \longrightarrow H^i_{2,\gamma}(\mathbb{R})) \cong 0 \qquad i=0,1$$

$$\cong \mathcal{H}^i_2(\mathbb{R})$$

so we get by [illegible]

$$\mathcal{H}^i_2(\mathbb{R}) \cong \mathrm{Im}\,(H^i_0(\mathbb{R}) \to H^i(\mathbb{R}))$$

<u>Thm</u> (APS) If M is a manifold with cylindrical end $N \times \mathbb{R}^+$ with the product metric then

$$\mathcal{H}^i_2(M,g) \cong \mathrm{Im}\,(H^i_0(M) \longrightarrow H^i(M)).$$

This is just an [illegible]

We can give a [illegible] more rigorous explanation by shifting the Fourier transform

Remember we considered

$$\frac{d}{dx} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$ via F.T where

$$\widehat{\tfrac{d}{dx}f}(\xi) = -i\xi\hat{f}(\xi)$$

So idea was that to invert we'd take

$$\left(\frac{1}{-i\xi}\cdot\hat{g}(\xi)\right)^{\vee}$$

But the problem is that at $\xi = 0$, this is not integrable (even if $g \in C_0^\infty$).

However, if we consider $g \in e^{\gamma x}L^2$ then

$$g = e^{\gamma x}\underbrace{e^{-\gamma x}g(x)}_{\in L^2}$$

so $$g = e^{\gamma x}\left(e^{-\gamma x}g(x)\right)^{\wedge\vee} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\gamma x}e^{ix\xi}\int_{\mathbb{R}} e^{-iy\xi}e^{-\gamma y}g(y)\,dy\,d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)(\xi-i\gamma)}g(y)\,dy\,d\xi$$

So this shifts the FT from integrating over $\mathrm{Re} \subset \mathbb{C}$ to over $\mathrm{Re} - i\gamma \subset \mathbb{C}$

Call the transform on $e^{\gamma x} L^2$ $\wedge_\gamma$

Then we get

$$\widehat{\frac{d}{dx} f(x)}^{\gamma} = (\xi - i\gamma)\, \widehat{f(x)}^{\gamma}$$

if $\gamma \neq 0$, we <u>can</u> always use this to solve $\frac{d}{dx} f(x) = g(x)$:

via $\left(\frac{1}{\xi - i\gamma}\, \hat{g}^{\gamma}(\xi)\right)^{\vee_\gamma} = f$.

Shifting the transform to act on different weights is one idea underlying the calculus of Melrose.

More general versions of this calculus, constructed for more general noncpt geometries than cylinders, can be used to prove other analogues of the Hodge Thm. Similar to APS.

3:30 - 4:30 pm Aug 29, 2008

Introduction to Hodge Theory

Equates two numbers associated to a smooth Compact mfd.

<u>Thm</u>: (Hodge) If M is smooth, Compact.
dim of harmonic forms on M
= dim of de Reham cohomology on M.

$$\mathcal{H}^*(M) \cong H^*(M)$$

<u>Recall</u>, on $\mathbb{R}^2$

$$0 \xrightarrow{d} C^\infty(M) \xrightarrow{d^0} C^\infty(M, \Lambda^1) \xrightarrow{d^1} C^\infty(M, \Lambda^2) \xrightarrow{d^2} 0$$

$$d(f) = f_x\, dx + f_y\, dy$$
$$d(f\,dx + g\,dy) = (g_x - f_y)\, dx \wedge dy$$
$$d(df) = (\partial_x \partial_y f - \partial_y \partial_x f)\, dx \wedge dy = 0$$

$d^2 = 0$ in each degree
$\mathrm{Im}(d^i) \subset \ker(d^{i+1})$ for each i.

Furthermore, each $C^\infty(M, \Lambda^i)$ is a vector and d is a linear map. So $\mathrm{Im}(d^i) \subset \ker(d^{i+1})$ is a Containment of vector spaces. So one can form ~~the~~ quotient space

$$\frac{\ker d^{i+1}}{\mathrm{Im}\, d^i} = H^{i+1}(\mathbb{R}^2) = (i+1)\text{st DeRham cohomology}$$

of $M = \mathbb{R}^2$

define $\mathcal{H}^i(M, g) = \{\alpha \in C^\infty(M, \Lambda^i) : d\alpha = 0 = d^*\alpha\}$

$= \{\alpha \in C^\infty(M, \Lambda^i) : (d + d^*)\alpha \equiv 0\}$

$\mathcal{H}^0(S^1, g) = \{\alpha \in C^\infty(S^1) \mid d\alpha = d^*\alpha = 0\}$

$= \{\text{Constants}\}$

$\cong \mathbb{R}$

$\mathcal{H}^1(S^1, g) = \{\alpha \in C^\infty(S^1, \Lambda^1) \mid d\alpha = d^*\alpha = 0\}$

$0 \to C^\infty(S^1) \to C^\infty(S^1, \Lambda^1)$

d^* on 1-forms on S^1 is just

$d^*(g(\theta)d\theta) = -g'(\theta)$

So ~~ker d^*~~ $\mathcal{H}^1(S^1, g) =$ Const 1-forms.

So $\mathcal{H}^*(S^1, g) \cong H^*(S^1, g)$

Now Consider if $M = R$.

$$\begin{tikzcd} 0 \arrow[r] & C^\infty(R) \arrow[r, shift left, "d"] & C^\infty(R, \Lambda^1) \arrow[l, shift left, "d^*"] \arrow[r] & 0 \end{tikzcd}$$

$df = f'dx$

$d^*(f dx) = -f'$

$\mathcal{H}^0(R) = \{\text{constants}\} \cong R$

$\mathcal{H}^1(R) = \{\text{Constants} \cdot dx\} \cong \mathbb{R}$

$H^0(R) =$ Constants

$H^1(R)$ = smooth 1-forms / derivatives of smooth functions

$= 0$

FTC $\Rightarrow$ $g(x)dx = d\left(\int_0^x g(t)dt\right)$

Maybe we did not set up things correctly

Clue: d^* is the "adjoint" of d

$\langle Av, w\rangle = \langle v, A^* w\rangle$

$\Rightarrow$ We need inner products. spaces

L^2 is an inner product space.

$\langle f, g\rangle = \int f\bar{g}\, dvol.$

So when we deal with the adjoint, we really should be working with L^2, not C^∞

$$\begin{tikzcd} 0 \arrow[r] & L^2(R) \arrow[r, shift left, "d"] & L^2(R, \Lambda^1) \arrow[l, shift left, "d^*"] \arrow[r] & 0 \end{tikzcd}$$

Now $\mathcal{H}^0_{L^2}(R, g) \cong \{0\} \neq H^0(R)$

$\mathcal{H}^1_{L^2}(R, g) \cong \{0\} \cong H^1(R)$

L^2-cohomology:

$$H^0_{L^2}(R) = \frac{\text{constant } L^2 \text{ functions}}{\{0\}} = 0 \cong \mathcal{H}^0_{L^2}(R)$$

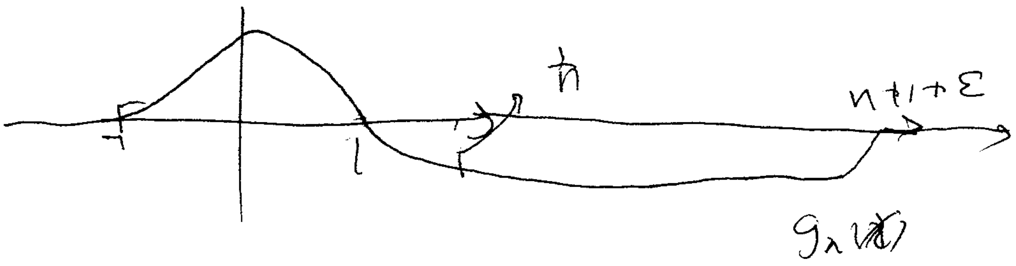
$$H^1_{L^2}(R) = \frac{\text{all } L^2\text{-functions}}{d(L^2\text{-functions})}$$

example:

$\int_R R\, dx = 1$

$$\int_{-\infty}^{x} g(t)\,dt = \begin{cases} 0 & x \le -1 \\ \vdots & \\ 1 & x > 1 \end{cases}$$

$[g(x)\,dx]$ is not the zero class in $H^1_{L^2}(R)$



Now $f_n(x) = \int_{-\infty}^{x} g_n(t)\,dt$

$$= \begin{cases} 0 & x < 1 \\ \vdots & \\ 1 & x = 1 \\ \vdots & \\ 0 & x > n+1+\varepsilon \end{cases} \in L^2$$

So $g_n(x)\,dx = d(f_n(x)) \quad \forall n.$

further

$$\|g - g_n\|_{L^2} = \int_1^{n+1+\varepsilon} g_n^2\,dx$$

$$= \int_1^{n+1+\varepsilon} \frac{1}{n^2} \cong \frac{n+1+\varepsilon}{n^2} \to 0$$

Thus $g \in \overline{\operatorname{Im} d}$

It turns out that

$$L^2(R;\Lambda^1) = \overline{\operatorname{Im} d}$$

Thus $\operatorname{Im} d$ is a dense subset of the kernel

$H^1_{L^2}(R)$ is ∞-dimensional & ugly.

Remember $\mathcal{H}^1_{L^2}(R) = \{0\} \cong \frac{\ker d}{\overline{\operatorname{Im} d}} = \overline{H^1_{L^2}(R)}$

the reduced L^2 de Rham cohomology

Thm: (?)

If M is a complete mfd. smooth. then

$$\mathcal{H}^i_{L^2}(M) \cong \overline{H}^i_{L^2}(M)$$

Problem: 1) Not a topological invariant

2) Does not necessarily satisfy M-V, etc.

The fact that Imd is not closed in $H^1_{L^2}(R)$ Computation is related to the fact that $d+d^*$ is not Fredholm on $L^2(R)$

So a different approach is to look at a different domain where $d+d^*$ is Fredholm. e.g. weighted L^2.

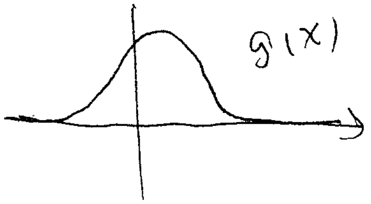
$$e^{\nu|x|}L^2(R) = \{ f \mid e^{-\nu|x|} f \in L^2 \}$$

Consider

$$0 \longrightarrow e^{\nu|x|}L^2(R) \xrightarrow{d} e^{\nu|x|}L^2(R\Lambda^1) \to 0$$

$$H^0_\nu(R) \cong \begin{cases} R & \nu > 0 \\ 0 & \nu < 0 \quad = \mathcal{H}^0 \end{cases}$$

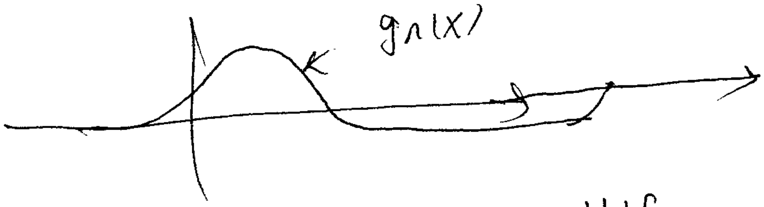
$$H^1_\nu(R) = \begin{cases} 0 & \nu > 0 \quad = \mathcal{H}^1 \\ R & \nu < 0 \end{cases}$$



suppose $\nu > 0$.

$$\int_{-\infty}^{x} g(t)\,dt = \begin{cases} 0 & x < -1 \\ 1 & x > 1 \end{cases}$$

suppose $\nu < 0$.



$$\|g - g_n\|_{L^2_\nu(\mathbb{R})} = \int_1^{n+1+\varepsilon} \phi(e^{-\nu|x|}|g - g_n||^2 \, dx$$

$$\to \infty \quad \text{as } n \to \infty$$

ϕ!

There is a natural map

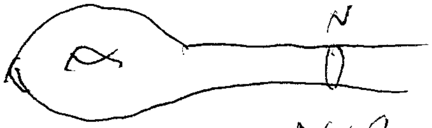
$$H^1_{-\nu}(\mathbb{R}) \to H^1_\nu(\mathbb{R})$$

$$\mathcal{H}^1_{L^2}(\mathbb{R}) \cong \operatorname{Im}(H^1_{-\nu}(\mathbb{R}) \to H^1_\nu(\mathbb{R}))$$

$$\cong \operatorname{Im}(H^1_{cpt}(\mathbb{R}) \to H^1(\mathbb{R}))$$

Thm (Atiyah - Patodi - Singer)

If M is a complete smooth mfd with a cylindrical end.



$N \times R_+$ with product metric

then $\mathcal{H}^i_{L^2}(M,g) \cong \operatorname{Im}\left(H^i_{cpt}(M) \to H^i(M)\right)$